

COMBINED EXPANSIONS OF PRODUCTS OF SYMMETRIC
POWER SUMS AND OF SUMS OF SYMMETRIC POWER
PRODUCTS WITH APPLICATION TO SAMPLING

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PREFACE

This article is divided into two parts. Part I has for its title "Combined Expansions of Products of Symmetric Power Sums and of Sums of Symmetric Power Products" and develops the general mathematical theory which is applied in Part II to "The Fundamentals of Sampling." Part II will appear in a latter issue of this journal.

Each part is treated as an organic unit and has its own introduction and bibliography. Each article is assigned a given number and each book is given a letter so that references can be indicated concisely in the body of the dissertation.

Each part is divided into chapters and sections. Braces are used to indicate the important formulas.

PART I. COMBINED EXPANSIONS OF PRODUCTS OF SYMMETRIC POWER SUMS AND OF SUMS OF SYMMETRIC POWER PRODUCTS

Introduction

The mathematical material which is presented here has proved useful in generalizing that portion of the fundamental theory of sampling in which relations are established between the moments of the sample and the moments of the parent population. It is the purpose to establish the theorems in algebraic form since they constitute an extension of partition and symmetric function theory and may be of value to someone not necessarily interested in sampling.

A great deal of work has been done in symmetric function theory but not much of this is of present value to the statistician. His problem deals with the "power sum" while the classical theory, for the most part, deals with the interrelations of elementary symmetric functions and monomial symmetric functions. Only one phase of the reasoning developed in this investigation seems to have received extensive consideration previously and that is the subject covered in Chapter III.

Previous authors have noted that much of symmetric function theory reduces, with a proper choice of notation, to partition theory. It is the plan of this treatise to present in Chapter I an outline of new partition theory which

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shows how the parts of one partition are combined to form the parts of another partition, and which serves as a means of expressing the main result of Chapters II, III, IV, V.

Chapter II shows how the formulas of Chapter I are applicable to the problem of finding products of power sums. The multiplication theorem for power sums, a generalization of the multinomial theorem, is stated in terms of power product sums and appropriate special cases are indicated.

Chapter III deals with the expansion of power product sums in terms of power sums and shows how the formulas of Chapter I may be used.

Chapter IV is the key chapter of the paper. The problem is to expand products of power sums in terms of power product sums, to multiply each power product sum by a quantity which is a uniquely defined function of the quantities composing the power product sum, and then to expand back in terms of all possible power sums. It is shown that the results can be written in a compact form which also utilizes the results of Chapter I. This result, as is shown in Part II, is directly applicable to the sampling problem of finding the moments of the sample moments in terms of the moments of the universe.

Extension is made to multivariate distributions in Chapter V.

Chapter I. The Combination of the Parts of Partitions

It is the purpose of this chapter to provide a precise notation which shows how the parts of one partition of r may be combined to form the parts of another partition of r . For example, 2111, a four part partition of 5, can be made into 32, a two part partition of 5, by combining the three unit parts into a new part or by combining the 2 with one of the unit parts to form the 3 and the other two unit parts to form the 2. This last formation can be made in three different ways since anyone of the unit parts might be combined with the 2. The combination of the parts of the partition 2111 to form the parts of the partition 32 is to be indicated symbolically by $P_{31} + 3P_{22}$ where the subscripts indicate the number of parts collected and the coefficients indicate the number of ways in which an equivalent collection can be made.

1. Definitions and Notation. a. *Partition* $[G; 105] [K; I; 1] [16; 105]$. We consider the integer r to be composed of r unit indistinguishable parcels and define the partitions of r to be all those different groupings into new parcels, each new parcel containing one or more unit parcels, such that each resultant grouping of parcels contains exactly all the original r unit parcels. For example the partitions of 4 are

$$4 ; 31 ; 22 ; 211 ; 1111$$

b. *Parts of Partitions*. The numbers of the grouped unit parcels indicate the parts of the partition. Thus the partitions of 4 above

$$4 ; 31 ; 22 ; 211 ; 1111 \text{ have respectively}$$

$$1 ; 2 ; 2 ; 3 ; 4 \text{ parts.}$$

The pattern 22 may also appear as 2^2 . In general a ρ part partition of r is to be designated by

$p_1 p_2 p_3 \dots p_\rho$ where the p 's may or may not be equal and where $p_1 + p_2 + p_3 + \dots + p_\rho = r$ or by

$$p_1^{\pi_1} \dots p_s^{\pi_s} \text{ where } \begin{cases} p_1 \asymp p_2 \asymp p_3 \asymp p_4 \asymp \dots \asymp p_s \\ p_1 \pi_1 + p_2 \pi_2 + \dots + p_s \pi_s = r \\ \pi_1 + \pi_2 + \dots + \pi_s = \rho \end{cases}$$

c. *Order of Partitions.* When the parts of a partition are arranged in descending order we say that the partition is ordered. Thus

$$p_1 p_2 \dots p_\rho \text{ is ordered if } p_1 \geq p_2 \geq p_3 \geq \dots \geq p_\rho$$

and $p_1^{\pi_1} \dots p_s^{\pi_s}$ is ordered if $p_1 > p_2 > p_3 > \dots > p_s$.

For example, 21^2 is an ordered partition while 312 is not. Unless otherwise specified it is hereafter assumed that all partitions are ordered.

It is sometimes convenient to refer to the order of the partition which is the size of the largest part, p_1 , when the partition is ordered. Thus the two part partition, 31, is of the order 3, while the four part partition, 1111, is of order 1. The set of the numbers $p_1^{\pi_1} \dots p_s^{\pi_s}$ is to be known as the complete order.

These definitions of order and part are consistent with the usual definitions. [16; 105-106] [K; I; 1] [G; 100]. The concept of complete order, as far as I know, is not found in the literature.

d. *Weight of Partitions. Isobaric Partitions.* The weight of any partition is defined to be the sum of all the parts of the partition. Thus the weight of $p_1^{\pi_1} \dots p_s^{\pi_s}$ is $p_1 \pi_1 + p_2 \pi_2 + \dots + p_s \pi_s = r$. Partitions having the same weight are called isobaric. Thus 4 and 211 are isobaric partitions.

e. *Algebraic Partitions.* If the r original units are composed of $a_1, a_2, a_3, \dots, a_r$ nonseparable primary units, then the result of combining these in any possible way is to be called an algebraic partition since the r original units are now replaced by the r algebraic quantities $a_1, a_2, \dots, a_r$. Thus a_1, a_2, a_3 may be combined to form

$$a_1 + a_2 + a_3; \overline{a_1 + a_2 \cdot a_3}; \overline{a_1 + a_3 \cdot a_2}; \overline{a_2 + a_3 \cdot a_1}; \overline{a_1 \cdot a_2 \cdot a_3}$$

which are the algebraic partitions of $a_1 + a_2 + a_3$.

The parts of the algebraic partitions are the resulting combinations while the order and complete order, which indicate the numbers of algebraic expressions combined, agree with the order and complete order of the partitions in which the a 's are unity. The weight, which is equal to the sum of the parts, is indicated by $w = a_1 + a_2 + \dots + a_r$. Thus if $a_1 = 5, a_2 = 4$, and $a_3 = 3$, $w = 12$. It is to be noted that the algebraic partitions are formed by combining the parts 5, 4, 3 and not by combining all parts of 12.

Now $a_1 a_2 \dots a_r$ is itself a partition of weight $w = a_1 + a_2 + \dots + a_r$. If groups of the a 's are alike it may be written

$$\alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \dots \alpha_h^{\alpha_h} \text{ where}$$

$$a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_h \alpha_h = w$$

$$\alpha_1 + \alpha_2 + \dots + \alpha_h = r.$$

Algebraic partitions having the same weight are called isobaric.

f. *Partition Combination Notation.* Let $\binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}}$ indicate the number of different ways the r units, ordinary or algebraic, can be collected to form the partition. Thus $\binom{1^5}{32}$ indicates the number of ways in which the five units can be collected to form a partition with three units in one part and two units in the other. Since the three units forming the first part can be selected in ${}_5C_3$ ways and since this selection automatically indicates the other two units forming the second part, it follows that

$$\binom{1^5}{32} = {}_5C_3 = {}_5C_2 = 10. \text{ It is to be noted that } \binom{1^4}{22} = 3 \approx {}_4C_2 = 6$$

for if the four unit parts are a_1, a_2, a_3, a_4 , then the three 22 partitions are

$$\overline{a_1 + a_2 \cdot a_3 + a_4}; \overline{a_1 + a_3 \cdot a_2 + a_4}; \overline{a_1 + a_4 \cdot a_2 + a_3}$$

since

$$\overline{a_3 + a_4 \cdot a_1 + a_2}; \overline{a_2 + a_4 \cdot a_1 + a_3}; \overline{a_2 + a_3 \cdot a_1 + a_4}$$

are essentially the same groupings as the first three indicated.

2. Formula for $\binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}}$. In establishing this formula we first take the case in which no part is repeated. i.e. $\pi_1 = \pi_2 = \dots = \pi_s = 1$ and $p_1 + p_2 + p_3 + \dots + p_s = r$. In this case the formula becomes

$$\binom{1^r}{p_1 p_2 \dots p_s} = \frac{r!}{p_1! p_2! \dots p_s!}$$

This results from the fact that the p_1 units can be grouped in ${}_rC_{p_1}$ different ways. The p_2 units then in ${}_{r-p_1}C_{p_2}$ different ways, the p_3 units then in ${}_{r-p_1-p_2}C_{p_3}$ different ways etc. So that

$$\begin{aligned} \binom{1^r}{p_1 p_2 \dots p_s} &= {}_rC_{p_1} \cdot {}_{r-p_1}C_{p_2} \cdot {}_{r-p_1-p_2}C_{p_3} \cdot \dots \cdot {}_{r-p_1-p_2-\dots-p_{s-1}}C_{p_s} \\ &= \frac{r!}{p_1! p_2! \dots p_s!} \quad \text{Compare [B; 49][19; 12]} \end{aligned}$$

If however $p_1 = p_2 = \dots = p_s$, then the same partition has been used $s!$ different times since $p_1, p_2, \dots, p_s$ may be interchanged in $s!$ different ways, so that

$$\binom{1^r}{p_1^s} = \frac{r!}{(p_1!)^s s!}$$

By similar reasoning

$$\binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}} = \frac{r!}{(p_1!)^{\pi_1} (p_2!)^{\pi_2} \dots (p_s!)^{\pi_s} \pi_1! \pi_2! \dots \pi_s!} \quad \{1\}$$

Compare [19; 12, 13] [I; II, 252]

3. Values of $\binom{a_1^{\alpha_1} a_2^{\alpha_2} \dots a_h^{\alpha_h}}{p_1^{\pi_1} p_2^{\pi_2} \dots p_s^{\pi_s}}$. The number of ways in which the r parts of $a_1^{\alpha_1} a_2^{\alpha_2} \dots a_h^{\alpha_h}$ may be collected to form the ρ parts of $p_1^{\pi_1} \dots p_s^{\pi_s}$ may be indicated by

$$\binom{a_1^{\alpha_1} a_2^{\alpha_2} \dots a_h^{\alpha_h}}{p_1^{\pi_1} p_2^{\pi_2} \dots p_s^{\pi_s}}. \text{ Thus } \binom{2111}{32} = 4 \text{ and } \binom{1111}{22} = 3$$

Formulas useful in evaluating this expression can be worked out from the results of this paper. A table of values of this expression for $w \leq 8$ has been given by the author [19; 29–32].

4. Notation for Combining the Parts of a Partition. Table I. We wish to indicate not only the number of ways in which a given r part partition of weight w can be grouped to form a ρ part partition of weight w , but also the number of parts of the r part partition grouped to form each of the ρ parts of the ρ part partition. As indicated in the opening paragraph, $P_{31} + 3P_{22} = P\binom{2111}{32}$ serves this purpose for the case in which the parts 2111 are collected to form 32. $P\binom{a_1^{\alpha_1} a_2^{\alpha_2} \dots a_h^{\alpha_h}}{p_1^{\pi_1} p_2^{\pi_2} \dots p_s^{\pi_s}}$ serves this purpose in the more general case. Its expansion gives sums of P functions whose subscripts are the numbers of parts combined and whose coefficients are the number of ways of forming the partitions from the parts. For example

$$P\binom{31}{4} = P_{21}; \quad P\binom{111}{21} = 3P_{21}; \text{ etc.}$$

The use of $P\binom{a_1^{\alpha_1} a_2^{\alpha_2} \dots a_h^{\alpha_h}}{p_1^{\pi_1} p_2^{\pi_2} \dots p_s^{\pi_s}}$ is so fundamental to the present approach that a table is provided showing the different values when $w \leq 6$.

Table I gives values of the function when $w = 1, 2, 3, 4, 5, 6$. The values $a_1^{\alpha_1} \dots a_h^{\alpha_h}$ are given in the left hand columns and the values of $p_1^{\pi_1} \dots p_s^{\pi_s}$ in the top row. The partitions are ordered from the top and from the left. To

TABLE I
 Values of $P \left(\begin{smallmatrix} a_1^{a_1} a_2^{a_2} \dots a_r^{a_r} \\ p_1^{p_1} p_2^{p_2} \dots p_s^{p_s} \end{smallmatrix} \right)$ when $W \leq 6$.

$W = 6$

$\frac{p}{a}$	6	51	42	33	411	321	222	31 ³	2 ² 1 ²	21 ⁴	1 ⁶
6	P_1										
51	P_2	P_{11}									
42	P_2		P_{11}								
33	P_2			P_{11}							
411	P_3	$2P_{21}$	P_{12}		P_{111}						
321	P_3	P_{21}	P_{21}	P_{12}		P_{111}					
222	P_3		$3P_{21}$				P_{111}				
31 ³	P_4	$3P_{31}$	$3P_{22}$	P_{13}	$3P_{211}$	$3P_{121}$		P_{1111}			
2 ² 1 ²	P_4	$2P_{31}$	$2P_{21}$	$2P_{22}$	P_{211}	$4P_{211}$	P_{112}		P_{1111}		
21 ⁴	P_5	$4P_{41}$	P_{41}	$4P_{32}$	$6P_{311}$	$4P_{311}$	$3P_{132}$	$4P_{2111}$	$6P_{1211}$	P_{11111}	
1 ⁶	P_6	$6P_{51}$	$15P_{42}$	$10P_{33}$	$15P_{411}$	$60P_{321}$	$15P_{222}$	$20P_{3111}$	$45P_{2211}$	$15P_{2111}$	P_{111111}

$W = 1$

$\frac{p}{a}$	1	P_1
1		

$W = 2$

$\frac{p}{a}$	2	11
2	P_1	
11	P_2	P_{11}

$W = 3$

$\frac{p}{a}$	3	21	111
3	P_1		
21	P_2	P_{11}	
111	P_3	$3P_{21}$	P_{111}

$W = 4$

$\frac{p}{a}$	4	31	22	211	1111
4	P_1				
31	P_2	P_{11}			
22	P_2		P_{11}		
211	P_3	$2P_{21}$	P_{12}	P_{111}	
1111	P_4	$4P_{31}$	$3P_{22}$	$6P_{211}$	P_{1111}

$W = 5$

$\frac{p}{a}$	5	41	32	311	221	2111	11111
5	P_1						
41	P_2	P_{11}					
32	P_2		P_{11}				
311	P_3	$2P_{21}$	P_{12}	P_{111}			
221	P_3	P_{21}	$2P_{21}$		P_{111}		
2111	P_4	$3P_{31}$	P_{31} $3P_{22}$	$3P_{211}$	$3P_{121}$	P_{1111}	
11111	P_5	$5P_{41}$	$10P_{32}$	$10P_{311}$	$15P_{221}$	$10P_{2111}$	P_{11111}

find a given value, say $P\left(\begin{smallmatrix} 2111 \\ 22 \end{smallmatrix}\right)$ we note that $w = 5$, look for 2111 on the left and 32 at the top. The result is $P_{31} + 3P_{22}$. In the table the order of the subscripts is important in indicating the number of parts collected to form the respective parts of the ordered $p_1^{\pi_1} \dots p_s^{\pi_s}$.

The values in the table previously mentioned [19; 29-32] may be obtained when $w \leq 6$ by placing every P in Table I equal to unity.

5. **Value of $P(a_1^{\alpha_1} a_2^{\alpha_2} \dots a_h^{\alpha_h})$.** The parts of the partition $a_1^{\alpha_1} a_2^{\alpha_2} \dots a_h^{\alpha_h}$ may be collected to form a large number of partitions of the type $p_1^{\pi_1} \dots p_s^{\pi_s}$. Thus the parts of the partition 2111 may be collected to form 5, 41, 32, 311, 221, 2111. We denote by $P(2111)$ the values

$$\begin{aligned} P\left(\begin{smallmatrix} 2111 \\ 5 \end{smallmatrix}\right) 5 + P\left(\begin{smallmatrix} 2111 \\ 41 \end{smallmatrix}\right) 41 + P\left(\begin{smallmatrix} 2111 \\ 32 \end{smallmatrix}\right) 32 + P\left(\begin{smallmatrix} 2111 \\ 311 \end{smallmatrix}\right) 311 + P\left(\begin{smallmatrix} 2111 \\ 221 \end{smallmatrix}\right) 221 \\ + P\left(\begin{smallmatrix} 2111 \\ 2111 \end{smallmatrix}\right) 2111 = P_4 5 + 3P_{31} 41 + [P_{31} + 3P_{22}] 32 + 3P_{211} 311 \\ + 3P_{211} 221 + P_{1111} 2111 \end{aligned}$$

and in general

$$P(a_1^{\alpha_1} a_2^{\alpha_2} \dots a_h^{\alpha_h}) = \sum P\left(\begin{smallmatrix} a_1^{\alpha_1} \dots a_h^{\alpha_h} \\ p_1^{\pi_1} \dots p_s^{\pi_s} \end{smallmatrix}\right) p_1^{\pi_1} \dots p_s^{\pi_s} \dots \{2\}$$

where the summation holds for every partition $p_1^{\pi_1} \dots p_s^{\pi_s}$ which can be formed by combining parts of $a_1^{\alpha_1} \dots a_h^{\alpha_h}$. The values of $P(a_1^{\alpha_1} \dots a_h^{\alpha_h})$ for $w \leq 6$ are given in the rows of Table I. Thus the value of $P(2111)$ above is found along the row 2111 where $w = 5$.

6. **Values of $P(1^r)$ and $P(a^r)$.** When $a_1 = 1$ and $\alpha_1 = r$ we have

$$P(1^r) = \sum P\left(\begin{smallmatrix} 1^r \\ p_1^{\pi_1} \dots p_s^{\pi_s} \end{smallmatrix}\right) p_1^{\pi_1} \dots p_s^{\pi_s}$$

and since there are $\left(\begin{smallmatrix} 1^r \\ p_1^{\pi_1} \dots p_s^{\pi_s} \end{smallmatrix}\right)$ different ways of forming $p_1^{\pi_1} \dots p_s^{\pi_s}$ from the r units and each way is indicated by $P_{p_1^{\pi_1} \dots p_s^{\pi_s}}$ we have

$$P(1^r) = \sum \left(\begin{smallmatrix} 1^r \\ p_1^{\pi_1} \dots p_s^{\pi_s} \end{smallmatrix}\right) P_{p_1^{\pi_1} \dots p_s^{\pi_s}} p_1^{\pi_1} \dots p_s^{\pi_s}. \quad \{3\}$$

When $r = 2, 3, 4$, etc., we get

$$P(1^2) = P_2 2 + P_{11} 1^2$$

$$P(1^3) = P_3 3 + 3P_{21} 21 + P_{111} 1^3$$

$$P(1^4) = P_4 4 + 4P_{31} 31 + 3P_{22} 22 + 6P_{211} 211 + P_{1111} 1^4$$

etc.

as indicated in Table I.

Similarly when $a_1 = a$ and $\alpha_1 = r \{2\}$ becomes

$$P(a^r) = \sum \binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}} P_{p_1^{\pi_1} \dots p_s^{\pi_s}} (ap_1)^{\pi_1} (ap_2)^{\pi_2} \dots (ap_s)^{\pi_s} \quad \{3'\}$$

since there are $\binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}}$ different ways of forming the partition $(ap_1)^{\pi_1} (ap_2)^{\pi_2} \dots (ap_s)^{\pi_s}$ from the r equal a 's and each way is indicated by $P_{p_1^{\pi_1} \dots p_s^{\pi_s}}$.

For example

$$P(a) = P\left(\begin{smallmatrix} a \\ a \end{smallmatrix}\right) a = P_1 a$$

$$P(a^2) = P\left(\begin{smallmatrix} a^2 \\ 2a \end{smallmatrix}\right) + P\left(\begin{smallmatrix} a^2 \\ aa \end{smallmatrix}\right) = P_2 2a + P_{11} a^2$$

$$P(a^3) = P\left(\begin{smallmatrix} a^3 \\ 3a \end{smallmatrix}\right) + P\left(\begin{smallmatrix} a \\ 2a \cdot a \end{smallmatrix}\right) + P\left(\begin{smallmatrix} a \\ a \cdot a \cdot a \end{smallmatrix}\right) = P_3 3a + 3P_{21} 2a \cdot a + P_{111} a^3$$

$$P(a^4) = P_4 4a + 4P_{31} 3a \cdot a + 3P_{22} 2a \cdot 2a + 6P_{211} 2a \cdot a^2 + P_{1111} a^4.$$

7. Values of $P(a_1 a_2 \dots a_r)$. From the definition

$$P(a_1) = P\left(\begin{smallmatrix} a_1 \\ a_1 \end{smallmatrix}\right) a_1 = P_1 a_1$$

$$\begin{aligned} P(a_1 a_2) &= P\left(\begin{smallmatrix} a_1 a_2 \\ a_1 + a_2 \end{smallmatrix}\right) + P\left(\begin{smallmatrix} a_1 a_2 \\ a_1 a_2 \end{smallmatrix}\right) \\ &= P_2 \overline{a_1 + a_2} + P_{11} a_1 a_2 \end{aligned}$$

$$\begin{aligned} P(a_1 a_2 a_3) &= P\left(\begin{smallmatrix} a_1 a_2 a_3 \\ a_1 + a_2 + a_3 \end{smallmatrix}\right) (a_1 + a_2 + a_3) + P\left(\begin{smallmatrix} a_1 a_2 a_3 \\ a_1 + a_2 \cdot a_3 \end{smallmatrix}\right) (\overline{a_1 + a_2 \cdot a_3}) \\ &\quad + P\left(\begin{smallmatrix} a_1 a_2 a_3 \\ a_1 + a_3 \cdot a_2 \end{smallmatrix}\right) (\overline{a_1 + a_3 \cdot a_2}) + P\left(\begin{smallmatrix} a_1 a_2 a_3 \\ a_2 + a_3 \cdot a_1 \end{smallmatrix}\right) (\overline{a_2 + a_3 \cdot a_1}) \\ &\quad + P\left(\begin{smallmatrix} a_1 a_2 a_3 \\ a_1 a_2 a_3 \end{smallmatrix}\right) (a_1 a_2 a_3) = P_3 (a_1 + a_2 + a_3) \\ &\quad + P_{21} \{ (\overline{a_1 + a_2 \cdot a_3}) + (\overline{a_1 + a_3 \cdot a_2}) + (\overline{a_2 + a_3 \cdot a_1}) \} + P_{111} (a_1 a_2 a_3) \\ &\quad \text{etc.} \end{aligned}$$

Now if complete order of the general partition indicates the number of a 's collected to form the partition, the subscripts of the P 's are the respective complete orders. If we indicate the sum of partitions having the same complete order by the term "partition type" and indicate the partition type composed of all terms having the same complete order

$$p_1^{\pi_1} \dots p_s^{\pi_s} \text{ by } T_{p_1^{\pi_1} \dots p_s^{\pi_s}}$$

then

$$P(a_1) = P_1 T_1$$

$$P(a_1 a_2) = P_2 T_2 + P_{11} T_{11}$$

$$P(a_1 a_2 a_3) = P_3 T_3 + P_{21} T_{21} + P_{111} T_{111}$$

$$P(a_1 a_2 a_3 a_4) = P_4 T_4 + P_{31} T_{31} + P_{22} T_{22} + P_{211} T_{211} + P_{1111} T_{1111}$$

etc.

and in general

$$P(a_1 a_2 \dots a_r) = \sum P_{p_1^{\pi_1} \dots p_s^{\pi_s}} T_{p_1^{\pi_1} \dots p_s^{\pi_s}} \quad \{4\}$$

This formula can be used in writing the formula of Table II or formulas of weight greater than 6. Thus

$$P(543) \text{ is given by } P_3 T_3 + P_{21} T_{21} + P_{111} T_{111}$$

where

$$T_3 = 12, T_{21} = 9 \cdot 3 + 8 \cdot 4 + 7 \cdot 5, \text{ and } T_{111} = 5 \cdot 4 \cdot 3$$

where the dots do not indicate multiplication, but merely the separation of the parts.

In general $T_{p_1^{\pi_1} \dots p_s^{\pi_s}}$ is composed of $\binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}}$ partitions since this is the number of ways in which the a 's can be combined to form partitions having the same complete order, $p_1^{\pi_1} \dots p_s^{\pi_s}$.

Formula $\{3'\}$ is a special case of this formula. If the a 's are all equal, the $\binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}}$ partitions are equal so that $T_{p_1^{\pi_1} \dots p_s^{\pi_s}} = \binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}} (ap_1)^{\pi_1} \dots (ap_s)^{\pi_s}$. Substitution in $\{4\}$ gives $\{3'\}$. Similarly $\{4\}$ gives $\{3\}$ when all the a 's are unity.

8. Generalization from Symmetry. The function $P(a_1 a_2 \dots a_r)$ is a symmetric function of the parts $a_1, a_2, \dots, a_r$, i.e., the interchange of any two of the parts does not change the value of the function. It is possible to use this fact as a basis of generalization and to derive $\{4\}$ from $\{3\}$ by its use. From $\{3\}$ we have

$$P(1^r) = \sum \binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}} P_{p_1^{\pi_1} \dots p_s^{\pi_s}} p_1^{\pi_1} \dots p_s^{\pi_s} \quad \{3\}$$

where $\binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}}$ is the number of the equivalent partitions which can be formed from the r units. In case the r units are replaced by the r different a 's, there will result $\binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}}$ different partitions having the same complete

order. These $\binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}}$ different partitions defined by $T_{p_1^{\pi_1} \dots p_s^{\pi_s}}$ replace the $\binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}}$ equivalent partitions of $\{2\}$ and we have

$$P(a_1 a_2 \dots a_r) = \sum P_{p_1^{\pi_1} \dots p_s^{\pi_s}} T_{p_1^{\pi_1} \dots p_s^{\pi_s}} \quad \{4\}$$

9. The Recursion Rule. It is possible to establish a recursion property by which the value of $P(a_1 a_2 \dots a_r a_{r+1})$ can be obtained from the value of $P(a_1 a_2 \dots a_r)$. We note, from the results of Table I or by $\{4\}$ that

$$P(3) = P_1 3$$

$$P(32) = P_2 5 + P_{11} 32$$

$$P(321) = P_3 6 + P_{21} 51 + P_{21} 42 + P_{12} 33 + P_{111} 111$$

$P(32)$ is obtained from $P(3)$ by symbolic multiplication of its expansion, $P_1(3)$, by the expansion of $P(2)$, $P_1(2)$. This symbolic multiplication is accomplished by adding the 2 to the 3 and also suffixing the 2 to the 3. If the 2 is added, the subscripts of the P 's are added while if suffixed, the subscripts of the P 's are suffixed.

More generally if $P(a_1) = P_1(a_1)$ and $P(a_2) = P_1(a_2)$, then the result $P(a_1 a_2) = P_2(\overline{a_1 + a_2}) + P_{11}(a_1) (a_2)$ is obtained by multiplying $P_1(a_1)$ by $P_2(a_2)$ [or $P_2(a_2)$ by $P_1(a_1)$] symbolically if the subscripts are added when the a 's are added and suffixed when the a 's are suffixed. Similarly $P(a_1 a_2) = P_2(\overline{a_1 + a_2}) + P_{11}(a_1) (a_2)$ when multiplied by $P(a_3) = P_1(a_3)$ gives

$$\begin{aligned} P(a_1 a_2 a_3) = & P_3(\overline{a_1 + a_2 + a_3}) + P_{21}(\overline{a_1 + a_2} \cdot a_3) + P_{21}(\overline{a_1 + a_3} \cdot a_2) \\ & + P_{12}(\overline{a_1 \cdot a_2 + a_3}) + P_{111}(a_1 a_2 a_3) \end{aligned}$$

when the rule of multiplication is the adding of a_3 in turn to every part of every partition with the appropriate adding of subscripts and the suffixing of a_3 to every partition with the corresponding suffixing of subscripts. It is important to note that the P coefficient of $a_1 \cdot a_2 + a_3$ is P_{12} and not P_{21} although the term could be written $P_{21} a_2 + a_3 \cdot a_1$. The applications do not demand the retention of a given order of subscripts though the continued application of the recursion rule does demand it.

In general the value of $P(a_1 \dots a_r a_{r+1})$ can be obtained from the value of $P(a_1 a_2 \dots a_r)$ by the symbolic multiplication of the expansion $P(a_1 a_2 \dots a_r)$ by $P_1(a_{r+1})$ since all possible algebraic partitions of $a_1 + a_2 + \dots + a_r + a_{r+1}$ are obtained from all possible algebraic partitions of $a_1 + a_2 + \dots + a_r$ by adding a_{r+1} in turn to each part of each partition and by suffixing it to each partition. The corresponding P subscript, indicating the number of a 's collected, is increased by 1.

The recursion rule is useful in checking the entries of Table I. As a matter

of fact Table I was computed with its use and the order of the subscripts is that which results from its use. The rule is also useful in finding values when $w > 6$. For example, since

$$P(321) = P_36 + P_{21}51 + P_{21}42 + P_{12}33 + P_{111}321$$

$$\begin{aligned} P(3221) &= P_48 + P_{31}62 + P_{31}71 + P_{22}53 + P_{211}512 + P_{31}62 + P_{22}44 \\ &\quad + P_{211}422 + P_{22}53 + P_{13}35 + P_{121}332 + P_{211}521 + P_{121}341 + P_{112}323 \\ &\quad + P_{1111}3212 = P_48 + P_{31}71 + 2P_{31}62 + (P_{31} + 2P_{22})53 + P_{22}44 \\ &\quad + 2P_{211}521 + P_{211}431 + P_{211}422 + 2P_{211}332 + P_{1111}3221. \end{aligned}$$

A useful check is based on the fact that the sum of the P coefficients of $P(a_1 \dots a_r)$ should equal the sum of the coefficients of $P(1^r)$. In the above illustration the sum of the coefficients is $P_4 + 4P_{31} + 3P_{22} + 6P_{211} + P_{1111}$ as desired.

10. Use of the P Function Formulas. The P function formulas, as defined, represent concisely the ways in which the parts of a given partition may be combined to get the parts of other partitions. They are also useful in writing expansions of certain partition functions whose expanded values are expressed in terms of other partition functions. They are used, in this paper, in expressing the multinomial theorem, the multiplication theorem for power sums, the expansions of power product sums in terms of power sums, expansions of monomial symmetric functions in terms of power sums, the double expansion theorem itself, the coefficients in the double expansion theorem as well as the sampling laws of Part II. They are also useful in representing the expansions of different moment functions and can be associated with important concepts of mathematics and statistics such as, for example, the differences of 0. Such applications, however, are not pertinent to the line of reasoning which is developed in Chapters II, III, IV, V.

Chapter II

It is the purpose of this chapter to obtain formulas for the expansion of power sums.

11. Definitions. a. *Power Sum.* Let x be a variable which is restricted to the N variates, $x_1, x_2, x_3, \dots, x_N$. Then the a -th power sum of the variable indicated by (a) is defined to be

$$(a) = x_1^a + x_2^a + \dots + x_N^a = \sum_{i=1}^N x_i^a \quad \{5\}$$

It is assumed for the purposes of this paper that a is a positive integer or 0.

b. *Power Product Sum.* The expression $\sum_{i \neq j} x_i^{a_1} x_j^{a_2}$ is to be called a power

product sum since it is composed of the sum of products of the powers of the variates. It is to be denoted by $(a_1 \cdot a_2)$ or $(a_1 a_2)$. Thus $\sum_{i \neq j} x_i^3 x_j^2 = (3 \cdot 2)$ or (32). The value $(a \cdot a) = (a^2) = \sum_{i \neq j} x_i^a x_j^a$ is a special case of $(a_1 a_2)$ where $a_2 = a_1 = a$. In general the power product sum is defined by the right hand member and indicated by the left hand member of

$$(a_1 a_2 \cdots a_r) = \sum_{i_1 \neq i_2 \neq i_3 \neq \cdots \neq i_r} x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_r}^{a_p} \quad \{6\}$$

If $i_1 = i_2$, the power product sum becomes

$$(\overline{a_1 + a_2 \cdot a_3 \cdot a_4 \cdots a_r}) = \sum_{i_1 = i_2 \neq i_3 \neq i_4 \neq \cdots \neq i_r} x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_p}^{a_p} \quad \{7\}$$

There are many different definitions since there are many different ways of indicating equality relations among the i 's. Each results in a unique power product sum which is to be called, for brevity, a power product. If the a 's are all unity, there are many duplicates. Thus for the grouping $p_1^{\pi_1} \cdots p_s^{\pi_s}$, there are $\binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ equal power products $(p_1^{\pi_1} \cdots p_s^{\pi_s})$. In the more general case we can let $T_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ represent the $\binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ different power products having the same complete order, $p_1^{\pi_1} \cdots p_s^{\pi_s}$. We may represent any one of these forms having this complete order by

$$(q_1 q_2 q_3 \cdots q_\rho)$$

where

$$q_1 + q_2 + q_3 + \cdots + q_\rho = w$$

or by

$$(q_1^{x_1} q_2^{x_2} \cdots q_t^{x_t})^*$$

where

$$q_1 x_1 + q_2 x_2 + \cdots + q_t x_t = w$$

and

$$x_1 + x_2 + \cdots + x_t = \rho.$$

c. *Symmetric Functions.* Both the power sum and the power product are symmetric functions of the variates since the interchange of any x_i with any x_j does not change the value of the function. Also the powder product having ρ parts is composed of $N^{(\rho)}$ products of powers since the first group of equal i 's may be selected in N ways, the next group in $N - 1$ ways etc.

d. *Monomial Symmetric Function.* It is customary to use the monomial symmetric function which is defined as

$$\sum_{i_1 < i_2 < i_3 < \cdots < i_\rho} x_{i_1}^{q_1} x_{i_2}^{q_2} \cdots x_{i_\rho}^{q_\rho}$$

* "It was intended that the letter representing the exponents of the q 's should be the Greek 'chi,' and not the English 'x.'"

and which we designate by $M(q_1 \dots q_\rho)$ or by $M(q_1^{x_1} \dots q_t^{x_t})$. This function is not useful for our purposes since the number of terms in its expansion varies with the number of repeated q 's. For example if $N = 3$ and $q_1 \approx q_2$; $M(q_1 q_2) = x_1^{q_1} x_2^{q_2} + x_1^{q_1} x_3^{q_2} + x_2^{q_1} x_3^{q_2} + x_2^{q_1} x_1^{q_2} + x_3^{q_1} x_1^{q_2} + x_3^{q_1} x_2^{q_2} = (q_1 q_2)$ while if $q_1 = q_2 = q$

$$M(q^2) = x_1^q x_2^q + x_1^q x_3^q + x_2^q x_3^q = \frac{(q^2)}{2!}.$$

The monomial symmetric function keeps the number of product terms a minimum by eliminating all repeated terms while the power product sum keeps the number of product terms the same by the use of repeated terms, when some of the parts are alike.

12. The Formula Connecting $(q_1^{x_1} q_2^{x_2} \dots q_t^{x_t})$ and $M(q_1^{x_1} \dots q_t^{x_t})$. The power product is composed of $N^{(\rho)}$ products, each of which is repeated $x_1! x_2! \dots x_t!$ times. The monomial symmetric function is composed of the $\frac{N^{(\rho)}}{x_1! x_2! \dots x_t!}$ different products which, when repeated $x_1! x_2! \dots x_t!$ times, gives the $N^{(\rho)}$ terms above. Hence

$$(q_1^{x_1} \dots q_t^{x_t}) = x_1! x_2! \dots x_t! M(q_1^{x_1} \dots q_t^{x_t}) \quad \{8\}$$

$$M(q_1^{x_1} \dots q_t^{x_t}) = \frac{1}{x_1! x_2! \dots x_t!} (q_1^{x_1} \dots q_t^{x_t}) \quad \{9\}$$

In the special case in which $q_1 = 1$ and $x_1 = \rho$

$$(1^\rho) = \rho! M(1^\rho) \quad \text{and} \quad M(1^\rho) = \frac{(1^\rho)}{\rho!} \quad \{10\}$$

The function, $M(1^\rho)$ is commonly called an elementary symmetric function. We refer to the corresponding (1^ρ) as the unitary power product sum.

13. Correspondence of Partitions and Power Products. To each power product $(q_1^{x_1} \dots q_t^{x_t})$ there corresponds an algebraic partition $q_1^{x_1} \dots q_t^{x_t}$ having ρ parts and weight $w = a_1 + a_2 + \dots + a_r$.

This follows at once from the definitions and notation. Thus if $w = a_1 + a_2 + a_3$, the power product

$$\sum_{i_1=i_2 \approx i_3} x_{i_1}^{a_1} x_{i_2}^{a_2} x_{i_3}^{a_3} = \sum_{i_1 \approx i_2} x_{i_1}^{a_1+a_2} x_{i_2}^{a_3} = \overline{(a_1 + a_2 \cdot a_3)}$$

is, by notation, associated with the partition $\overline{a_1 + a_2 \cdot a_3}$. Conversely each algebraic partition, when enclosed in parentheses, represents a power product sum.

This proposition is useful in that it enables one to establish a relationship between the theory of power product sums and the theory of partitions. Earlier writers have used a similar correspondence in relating the theory of monomial

symmetric functions to that of partitions. See for instance [3; 106], [4; 5] [5; I; 7].

Due to this correspondence we do not hesitate to apply such terms as part, order, complete order, similar, etc. to the power product as well as to the algebraic partition. Also the sum of all power products $(q_1^{x_1} \dots q_s^{x_s})$ having the same complete order is represented by $T(p_1^{x_1} \dots p_s^{x_s})$. This represents the sum of $\binom{1^r}{p_1^{x_1} \dots p_s^{x_s}}$ similar power products.

14. The Multiplication Theorem for Power Sums. The correspondence property enables us to derive a theorem, to be known as the multiplication theorem, which expresses products of power sums in terms of power products. The type of argument is introduced by establishing simple cases of the theorem

$$\begin{aligned} (a_1)(a_2) &= \left(\sum_{i=1} x_i^{a_1} \right) \left(\sum_{i=1} x_i^{a_2} \right) = \sum_{\substack{i_1=1 \\ i_2=1}}^N x_{i_1}^{a_1} x_{i_2}^{a_2} \\ &= \sum_{i_1=i_2} x_{i_1}^{a_1} x_{i_2}^{a_2} + \sum_{i_1 \neq i_2} x_{i_1}^{a_1} x_{i_2}^{a_2} = (a_1 + a_2) + (a_1 a_2) \end{aligned}$$

$$\begin{aligned} (a_1)(a_2)(a_3) &= \left(\sum x_i^{a_1} \right) \left(\sum x_i^{a_2} \right) \left(\sum x_i^{a_3} \right) = \sum_{i_1, i_2, i_3} x_{i_1}^{a_1} x_{i_2}^{a_2} x_{i_3}^{a_3} \\ &= (a_1 + a_2 + a_3) + (\overline{a_1 + a_2} \cdot a_3) + (\overline{a_1 + a_3} \cdot a_2) + (\overline{a_1 \cdot a_2} \cdot a_3) + (a_1 \cdot a_2 \cdot a_3) \end{aligned}$$

since the value $\sum_{i_1, i_2, i_3}$ is broken into

$$\sum_{i_1=i_2=i_3}, \quad \sum_{i_1=i_2 \neq i_3}, \quad \sum_{i_1=i_3 \neq i_2}, \quad \sum_{i_1 \neq i_2=i_3}, \quad \sum_{i_1 \neq i_2 \neq i_3}.$$

In general, when $r \leq N$

$$(a_1)(a_2) \dots (a_r) = \sum_{i_1, i_2, \dots, i_r} x_{i_1}^{a_1} x_{i_2}^{a_2} \dots x_{i_r}^{a_r}$$

and this can be broken into summations featuring different equality relations. These summations define all the different power product sums of weight $w = a_1 + a_2 + \dots + a_r$. The different algebraic partitions of $a_1 + a_2 + a_3 + \dots + a_r$ correspond to the different power product sums. It follows at once that the value of $(a_1)(a_2) \dots (a_r)$ is obtained by writing each algebraic partition of $a_1 + a_2 + \dots + a_r$, enclosing it in parentheses to represent a power product, and adding. More symbolically we have

$$(a_1)(a_2) \dots (a_r) = \sum (q_1^{x_1} \dots q_t^{x_t}) \quad \{11\}$$

where $q_1^{x_1} \dots q_t^{x_t}$ represents any algebraic partition of $a_1 + a_2 + \dots + a_r$ and the summation holds for all such partitions or by

$$(a_1)(a_2) \dots (a_r) = \sum T(p_1^{x_1} \dots p_s^{x_s}) \quad \{12\}$$

where $T(p_1^{x_1} \dots p_s^{x_s})$ represents the $\binom{1^r}{p_1^{x_1} \dots p_s^{x_s}}$ similar power products and the summation holds for each different complete order.

For example $(a_1)(a_2)(a_3) = T(3) + T(21) + T(111)$
 and $T(3) = (a_1 + a_2 + a_3)$, $T(21) = (\overline{a_1 + a_2 \cdot a_3}) + (\overline{a_1 + a_3 \cdot a_2}) + (\overline{a_2 + a_3 \cdot a_1})$,
 and $T(111) = (a_1 \cdot a_2 \cdot a_3)$.

The theorem has been established on the assumption that $r \leq N$. If such is not the case it is possible to satisfy the assumption by adding additional variates, x_{N+1} , x_{N+2} , $\dots$ x_r , all 0, without changing the value of the power sums or of the product of the power sums since the added terms are always 0. Thus

$$(x_1^a + x_2^a)(x_1^b + x_2^b)(x_1^c + x_2^c) = (x_1^a + x_2^a + x_3^a)(x_1^b + x_2^b + x_3^b)(x_1^c + x_2^c + x_3^c)$$

when $x_3 = 0$

Then

$$(a)(b)(c) = (a + b + c) + (\overline{a + b \cdot c}) + (\overline{a + c \cdot b}) + (\overline{b + c \cdot a}) + (a \cdot b \cdot c)$$

which is

$$\sum x_i^{a+b+c} + \sum_{i \succcurlyeq j} x_i^{a+b} x_j^c + \sum_{i \succcurlyeq j} x_i^{a+c} x_j^b + \sum_{i \succcurlyeq j} x_i^{b+c} x_j^a + \sum_{i \succcurlyeq j \succcurlyeq k} x_i^a x_j^b x_k^c$$

The term $\sum_{i \succcurlyeq j \succcurlyeq k} x_i^a x_j^b x_k^c = 0$ since every product composing it contains an $x_3 = 0$.

The other power product sums are to be applied to the original variates only since the terms involving x_3 are 0 in every case.

In general, if $r > N$, it is only necessary to write out the power product sums having N or less parts since all those having more than N parts will be 0.

15. The Multiplication Theorem Using the Results of Chapter I. Comparison of {12} with {4} shows that {12} can be obtained from {4} by placing $P(a_1 \dots a_r) = (a_1)(a_2) \dots (a_r)$, $T_{p_1^{\pi_1} \dots p_s^{\pi_s}} = T(p_1^{\pi_1} \dots p_s^{\pi_s})$ and $P_{p_1^{\pi_1} \dots p_s^{\pi_s}} = 1$. Since this can be done for all values of a and r it follows at once that the entire theory of Chapter I is applicable to the present problem. For example Table I shows that

$$P(321) = P_36 + P_{21}51 + P_{21}42 + P_{12}33 + P_{111}321$$

and it follows that

$$(3)(2)(1) = (6) + (51) + (42) + (33) + (321)$$

It should be noted that it is possible to use the table previously published [19; 29-32] since the entries in this table are the values obtained when $P_{p_1^{\pi_1} \dots p_s^{\pi_s}} = 1$. The value (3)(2)(1) may also be checked from this table.

16. The Multinomial Theorem. The multinomial theorem is a special case of the multiplication theorem for power sums in which the power sums are all equal. If $a_1 = a_2 = \dots = a_r = 1$,

$$T(p_1^{\pi_1} \dots p_s^{\pi_s}) = \binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}} (p_1^{\pi_1} \dots p_s^{\pi_s})$$

and {12} becomes

$$(1)^r = \sum \binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}} (p_1^{\pi_1} \dots p_s^{\pi_s}) \quad \{13\}$$

which is the multinomial theorem in terms of power product sums. Special cases are

$$(1)^2 = (2) + (11)$$

$$(1)^3 = (3) + 3(21) + (111)$$

$$(1)^4 = (4) + 4(31) + 3(22) + 6(211) + (1111)$$

etc.

The result of {13} may also be obtained immediately from {3} by placing $P(1^r) = (1)^r$, $P_{p_1^{\pi_1} \dots p_s^{\pi_s}} = 1$, and $p_1^{\pi_1} \dots p_s^{\pi_s} = (p_1^{\pi_1} \dots p_s^{\pi_s})$.

A more general form of the multinomial theorem is that in which $a_1 = a_2 = \dots = a_r = a$. In this case

$$(x_1^a + x_2^a + \dots + x_N^a)^r = (a)^r$$

and {12} gives

$$(a)^r = \sum \binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}} ((ap_1)^{\pi_1} \dots (ap_s)^{\pi_s}) \quad \{14\}$$

where $((ap_1)^{\pi_1} \dots (ap_s)^{\pi_s})$ has parts $ap_1, \dots, ap_s$. Thus

$$(a)^3 = (3a) + 3(2a \cdot a) + (a^3)$$

so that

$$(2)^3 = (6) + 3(4 \cdot 2) + (2^3).$$

The result {14} may also be obtained immediately from {3'} by placing $P(a^r) = (a)^r$, $P_{p_1^{\pi_1} \dots p_s^{\pi_s}} = 1$, and $(ap_1)^{\pi_1} \dots (ap_s)^{\pi_s} = ((ap_1)^{\pi_1} \dots (ap_s)^{\pi_s})$. When $N = 2$, {13} gives the binomial theorem

$$(1)^r = \sum \binom{1^r}{p_1 p_2} (p_1 p_2)$$

special cases of which are

$$(1)^2 = (2) + (11)$$

$$(1)^3 = (3) + 3(21)$$

$$(1)^4 = (4) + 4(31) + 3(22)$$

$$(1)^5 = (5) + 5(41) + 10(32)$$

etc.

These can be readily translated to the usual form. Thus

$$(a + b)^4 = a^4 + b^4 + 4(a^3b + b^3a^3) + 3(a^2b^2 + b^2a^2).$$

In a similar manner the trinomial theorem appears as

$$(1)^r = \sum \binom{1^r}{p_1 p_2 p_3} (p_1 p_2 p_3)$$

A special case of the multinomial theorem {13} is also useful in writing N^r in terms of sums of $N^{(\rho)}$. When the variates are all unity the power sums are all N , and the power product sums are the number of terms in the partition representing it. If a partition has ρ parts the number of terms in it is $N^{(\rho)}$. We then have

$$N^r = \sum \binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}} N^{(\rho)} \quad \{15\}$$

Special cases are

$$N^2 = N + N^{(2)}$$

$$N^3 = N + 3N^{(2)} + N^{(3)}$$

$$N^4 = N + 4N^{(2)} + 3N^{(2)} + 6N^{(3)} + N^{(4)} = N + 7N^{(2)} + 6N^{(3)} + N^{(4)}$$

etc.

17. The Use of Monomial Symmetric Functions. It is possible to express the results in terms of the monomial symmetric functions by means of {8}. Thus

$$\begin{aligned} (2)(2)(2) &= (6) + 3(42) + (222) \\ &= M(6) + 3M(42) + 6M(222). \end{aligned}$$

In general, Table I may be used to express products of power sums in terms of monomial symmetric functions. It is only necessary to place every $P_{p_1^{\pi_1} \dots p_s^{\pi_s}} = 1$ and to multiply by the factorials indicating the repeated entries at the head of each column. The table [19; 29-32] may be used similarly.

The multinomial theorem in terms of monomial symmetric functions becomes

$$(1)^r = \sum \binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}} \pi_1! \pi_2! \dots \pi_s! M(p_1^{\pi_1} \dots p_s^{\pi_s})$$

and by {1}

$$(1)^r = \sum \frac{r!}{(p_1!)^{\pi_1} (p_2!)^{\pi_2} \dots (p_s!)^{\pi_s}} M(p_1^{\pi_1} \dots p_s^{\pi_s}) \quad \{16\}$$

as it is conventionally stated.

18. The Multiplication Theorem from the Multinomial Theorem. It is possible to use generalization from symmetry in deriving the multiplication theorem from the multinomial theorem though this can not well be done from its conventional statement (16). The monomial symmetric function does not have the property that $M(a \cdot b) = M(a \cdot a)$ when $b = a$ while $(a \cdot b)$ does become $(a \cdot a)$ when $b = a$. The first step then is to reduce $\{16\}$ to power product sums by means of $\{9\}$. We then have

$$(1)^r = \sum \frac{r!}{(p_1!)^{\pi_1} (p_2!)^{\pi_2} \dots (p_s!)^{\pi_s}} \pi_1! \pi_2! \dots \pi_s! (p_1^{\pi_1} \dots p_s^{\pi_s})$$

Next it is necessary to introduce the factor $\binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}}$ for there are many equal terms for each value $p_1^{\pi_1} \dots p_s^{\pi_s}$ when the a 's are all unity. This is very easy in this case since the value of the coefficient of $(p_1^{\pi_1} \dots p_s^{\pi_s})$ is $\binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}}$. It follows at once that

$$(1)^r = \sum \binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}} (p_1^{\pi_1} \dots p_s^{\pi_s}).$$

Suppose that the r units are replaced by $a_1 a_2 \dots a_r$. Then the $\binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}}$ power products, $(p_1^{\pi_1} \dots p_s^{\pi_s})$ will be replaced by the $\binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}}$ different power products composing $T(p_1^{\pi_1} \dots p_s^{\pi_s})$. It follows at once that

$$(a_1)(a_2) \dots (a_r) = \sum T(p_1^{\pi_1} \dots p_s^{\pi_s}).$$

19. The Determination of the Coefficient of a Given Power Product in the Expansion of a Product of Power Sums. In some cases we wish to determine the coefficient of a given power product without computing the complete expansion. This is given by $P \binom{a_1^{\alpha_1} \dots a_h^{\alpha_h}}{q_1^{x_1} \dots q_t^{x_t}}$ where the P coefficients are unity. Thus the coefficient of (32) in the expansion of (2)(1)(1)(1) is found from $P \binom{2111}{32} = P_{31} + 3P_{22}$ and is 4.

20. Relation to Previous Results. The multiplication theorem may be viewed as a generalization of the multinomial theorem. A more general proof, applicable to multivariate problems, could be presented with the use of more involved notation. It seems wise rather to present the simpler one variate case and to emphasize the principle of generalization from symmetry which will enable us to write the multivariate laws with relative ease.

The general problem discussed here seems to have received a very small

amount of consideration as much of the extensive classical theory of symmetric functions is limited to the interrelations of the elementary symmetric functions and the monomial symmetric functions.

A monumental work on symmetric functions not subject to this limitation is the *Combinatory Analysis* of MacMahon [K]. MacMahon provided a technique for multiplying power sums in many variables as a special case of a more general theory. [K; II, 321].

Some of the work on alternants is closely related to the problem of products of power sums although the alternant, as usually defined, is limited to the case in which $r = N$ [I; II, 446]. For an example the reader is referred to a development by Muir [L; 335-6].

Thiele (1889) gave tables² of products of power sums in terms of monomial symmetric functions for partition products of weight ≤ 8 [H; 114-117]. J. R. Roe has later given one for $w \leq 10$ [N; Plates 17, 18]. Statisticians have sometimes stated the results in nontabular form. See for example, the multiplication formulas of Church [13; 81-83] [14; 370-1], whose results may not at first appear to agree with those above since Church has used a less compact notation and, of course, the monomial symmetric function.

The chief contributions of the present attack are

1. The use of the formulas and tables of Chapter I in writing expansions of products of power sums.
2. The use of power product sums in place of monomial symmetric functions which makes feasible.
3. Generalization from symmetry.

Chapter III

It is the purpose of this chapter to establish formulas giving the expansion of power products in terms of products of power sums.

21. The Binet (Waring) Identities. It is customary to introduce this subject with formulas for $M(a \cdot b)$, $M(a \cdot b \cdot c)$, etc. so we first derive the formulas for $(a \cdot b)$, $(a \cdot b \cdot c)$, etc. We may use the results of Chapter II since the problem here is the inverse of the multiplication problem. By the multiplication theorem

$$(a)(b) = (a + b) + (a \cdot b)$$

$$(a)(b)(c) = (a + b + c) + (\overline{a + b} \cdot c) + (\overline{a + c} \cdot b) + (\overline{b + c} \cdot a) + (a \cdot b \cdot c)$$

$$(a + b)(c) = (a + b + c) + (\overline{a + b} \cdot c)$$

² These tables are not accessible to me, but Thiele refers to them in his "Theory of Observations."

so we get

$$(a \cdot b) = (a)(b) - (a + b) \quad [17]$$

$$(a \cdot b \cdot c) = (a)(b)(c) - (a + b)(c) - (a + c)(b) - (b + c)(a) + 2(a + b + c) \quad \{18\}$$

Similarly

$$\begin{aligned}(a \cdot b \cdot c \cdot d) &= (a)(b)(c)(d) - (a + b)(c)(d) - (a + c)(b)(d) \\&\quad - (a + d)(b)(c) - (b + c)(a)(d) - (b + d)(a)(c) \\&\quad - (c + d)(a)(b) - (a + b)(c + a) - (a + c)(b + d) \\&\quad - (a + d)(b + c) + 2(a + b + c)(d) + 2(a + b + d)(c) \\&\quad + 2(a + c + d)(b) + 2(b + c + d)(a) - 6(a + b + c + d)\end{aligned}$$

... {19} ...

When $a \asymp b \asymp c \asymp d$, $\{18\}$, $\{19\}$, $\{20\}$ are also the formulas for $M(ab)$, $M(abc)$, $M(abcd)$. These formulas are quite commonly attributed to Binet who gave them in 1812 in connection with certain proofs of determinant theory [1; 284] [I; I; 81]. Waring should be given credit (see *Miscellanea Analytica* 1762). Binet gave no proof. The reader is also referred to the earlier work of Paoli [A; section 28].

A much more adequate treatment was given by Hirsch in the early 19th century [B; 35–38]. He wrote, out the terms for $M(a \cdot b \cdot c \cdot d \cdot e)$ and indicated a scheme for extending the results. More than this he proved that any “numerical expression”—his term for monomial symmetric function—can be reduced to numerical expression having one less part [B; 26]. The continued application of this theorem leads eventually to numerical expressions having only one part, i.e. to power sums. Hence all numerical expressions can be reduced to power sums [B; 27, 32].

Recent authors give essentially the same proof. See for example Bocher [J; 241–242] who states the theorem, “Every symmetric polynomial is a linear combination with constant coefficients of a certain number of the Σ ’s.” See also O’Toole [16; 114] and Burnside and Panton [E; 167]. Thus modern authors provide a proof of the fact that $M(a_1 \cdots a_r)$ can be expanded in terms of power sums but most of them fail to provide a formula giving this precise expansion. Even MacMahon after writing the values of $M(\lambda_\mu)$, $M(\lambda_{\mu r})$, $M(\lambda^2)$, $M(\lambda^3)$ avoids the immediate generalization by stating [K; I; 7], “In actual practice there are easier ways of calculating the many part functions and the general formula is of little importance.”

While MacMahon's statement has a certain amount of truth in that any given monomial symmetric function may be computed from others having one less part by the recursion property described by Hirsch, yet there are many cases in which a definite formula, rather than a method, is desirable. A formula

particularly is demanded by the statistician who is working with a large number of monomial symmetric functions simultaneously. See for example the remarks and efforts of Carver [15; 103–104, 119–120], Church [14; 373, 377–378], and O'Toole [16; 115].

Some authors have provided solutions and it appears that statisticians are not entirely familiar with all the work which has previously been done. It is the aim of the remainder of this chapter to suggest references which make previous work available to statisticians as well as to present a logical and quite complete development. The main results are not essentially new although their explicit statement in the language of power products is necessary for the development of the next chapter. The argument features the easy generalization from symmetry. The value of (1^r) is expressed in such a form that the value $(a_1 \cdots a_r)$ may be obtained immediately from it.

22. The Value of (1^r) from Waring's Expansion for the Elementary Symmetric Function. We first derive the formula (1^r) from the conventional Waring's expression for p_m in terms of the power sums. Burnside and Panton [E; II; 92] give this as

$$p_m = \sum \frac{(-1)^{r_1+r_2+\cdots+r_m} s_1^{r_1} s_2^{r_2} \cdots s_m^{r_m}}{\Gamma(r_1+1) \Gamma(r_2+1) \cdots \Gamma(r_m+1) 2^{r_2} 3^{r_3} \cdots m^{r_m}} \quad \{20\}$$

where $p_m = (-1)^m M(1^m)$ and where $S_1^{r_1} \cdots S_m^{r_m}$ is any $r_1 + r_2 + \cdots + r_m$ part partition of m . When $m = r$ and $p_1^{\pi_1} \cdots p_s^{\pi_s}$ is any ρ part partition of m , $\{20\}$ becomes

$$(-1)^r M(1^r) = \sum \frac{(-1)^\rho (p_1)^{\pi_1} \cdots (p_s)^{\pi_s}}{p_1!^{\pi_1} \cdots p_s!^{\pi_s} \pi_1! \pi_2! \cdots \pi_s!} \quad \{21\}$$

Dividing by $(-1)^r$ and noting that $(-1)^{\rho-r} = (-1)^{r-\rho}$ we have

$$M(1^r) = \sum \frac{(-1)^{r-\rho} (p_1)^{\pi_1} \cdots (p_s)^{\pi_s}}{p_1!^{\pi_1} \cdots p_s!^{\pi_s} \pi_1! \pi_2! \cdots \pi_s!} \quad \{22\}$$

and hence that

$$(1^r) = r! M(1^r) = \sum \frac{(-1)^{r-\rho} r! (p_1)^{\pi_1} \cdots (p_s)^{\pi_s}}{p_1!^{\pi_1} \cdots p_s!^{\pi_s} \pi_1! \pi_2! \cdots \pi_s!} \quad \{23\}$$

A second proof of $\{23\}$, given in the next sections, does not assume the formula $\{20\}$ and develops by easy stages. Although somewhat longer than the method above, it contacts much of the work that has been done in this field. It also provides two useful arithmetic checks dealing with the coefficients which the more analytic method above does not provide. Those who are familiar with $\{20\}$ above and are interested in the immediate development of the argument with the use of $\{23\}$ should turn to the equivalent $\{38\}$ of section 28.

23. The Newtonian Formulas. The development begins with the well known formulas connecting the power sums and the elementary symmetric functions which appeared in Newton's *Arithmetica Universalis*. These formulas are given by Bocher (J; 244) as follows

$$S_k - p_1 S_{k-1} + \cdots + (-1)^{k-1} p_{k-1} S_1 + (-1)^k k p_k = 0 \quad k = 1, 2, \dots \quad \{24\}$$

where S_k is the sum of the k -th powers and p is the i -th elementary symmetric function.

So many proofs of this theorem are accessible that a repetition here is hardly justifiable. A proof using calculus was given by Bocher (J; 243). Proofs using algebra only were given by Hirsch (B; 16) and Chrystal (F; I, 437). Muirhead (9; 66-70) gave three proofs of which the second is perhaps best adapted to the present development.

24. The Determinant Equivalent of (1'). It is usual to solve the Newtonian equations for the power sums (J; 244) but our objective is the solution in terms of the power sums. The equations are

$$\begin{aligned} p_1 &= (1) \\ (1)p_1 - 2p_2 &= (2) \\ (2)p_2 - (1)p_2 + 3p_3 &= (3) \\ (3)p_1 - (2)p_2 + (1)p_3 - 4p_4 &= (4) \\ &\vdots \end{aligned}$$

whence

$$p_r = \begin{vmatrix} 1 & 0 & 0 & \dots & 0 & (1) \\ (1) & -2 & 0 & \dots & 0 & (2) \\ (2) & -(1) & 3 & \dots & 0 & (3) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (r-2) & -(r-3) & (r-4) & \dots & (-1)^{r-2}(r-1) & (r-1) \\ (r-1) & -(r-2) & (r-3) & \dots & (-1)^{r-2}(1) & (r) \end{vmatrix}$$

$$p_r = \begin{vmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ (1) & -2 & 0 & \dots & 0 & 0 \\ (2) & -(1) & 3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (r-2) & -(r-3) & (r-4) & \dots & (-1)^{r-2}(r-1) & 0 \\ (r-1) & -(r-2) & (r-3) & \dots & (-1)^{r-2}(1) & (-1)^{r-1}r \end{vmatrix}$$

Next, factor out all the negative signs in the even numbered columns in each determinant. The number of these columns of negative signs is the same as

the number in the denominator if r is odd. If r is even, there is one more in the denominator. Hence the negative signs may be dropped in both determinants if $(-1)^{r-1}$ is inserted in the numerator. Furthermore the value of the determinant in the denominator is $r!$. Next, change the numerator by moving the r -th column to the first column position and inserting the compensating factor $(-1)^{r-1}$. If Δ_r represents the resulting numerator determinant, the value of p_r becomes

$$p_r = \frac{(-1)^{r-1}(-1)^{r-1}}{r!} \Delta_r$$

and

$$\Delta_r = r!p_r = (1^r).$$

We have then

$$(1^r) = \Delta_r = \begin{vmatrix} (1) & 1 & 0 & \dots & 0 & 0 \\ (2) & (1) & 2 & \dots & 0 & 0 \\ (3) & (2) & (1) & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ (r-1) & (r-2) & (r-3) & \dots & (1) & r-1 \\ (r) & (r-1) & (r-2) & \dots & (2) & (1) \end{vmatrix} \quad \{25\}$$

The determinant has received the attention of earlier writers {19; 3}. Generalizations of it will be mentioned at the close of the chapter. Its expansion in terms of power sums is known and may be written

$$\Delta_r = \sum \frac{(-1)^{r-\rho} r! (p_1)^{\pi_1} \dots (p_s)^{\pi_s}}{p_1!^{\pi_1} \dots p_s!^{\pi_s} \pi_1! \pi_2! \dots \pi_s!} \quad \{26\}$$

where

$$p_1 \pi_1 + p_2 \pi_2 + \dots + p_s \pi_s = r$$

and

$$\pi_1 + \pi_2 + \dots + \pi_s = \rho.$$

See for example O'Toole (16; 113).

It is at once evident that {26} is equivalent to {23}. Those who are familiar with the expansion of Δ_r above may wish to turn immediately to {38} of section 28 since the intervening sections are devoted to a rather detailed and rigorous expansion of the determinant. This development follows, in a general way, that given by Mola (5; 190-195).

25. The Expansion of $(1^r) = \Delta_r$. The determinant, Δ_r , is a special type of determinant which is known as a recurrent. There is a simple recursion property which is useful in its expansions in terms of products of power sums.

$$\Delta_{r+1} = \begin{vmatrix} (1) & 1 & 0 & \dots\dots\dots 0 & 0 \\ (2) & (1) & 2 & \dots\dots\dots 0 & 0 \\ (3) & (2) & (1) & \dots\dots\dots 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ (r) & (r-1) & (r-2) & \dots\dots\dots (1) & r \\ (r+1) & (r) & (r-1) & \dots\dots\dots (2) & (1) \end{vmatrix}$$

If we expand Δ_{r+1} in terms of the $(r+1)$ st column we have

$$\Delta_{r+1} = (1)\Delta_r - r\Delta_r \quad \{27\}$$

where Δ_r represents the determinant Δ_r with every power sum in the r -th row increased by unity. It is only necessary to arrive at some method of designating these terms if the above recurrence formula is to be applied. This can be done by inserting the power sum (1) before the other power sums which it is to multiply. Also in forming Δ_r add unity to the first power sums in the expansion of Δ_r being careful to retain the previous order. Thus

$$\left. \begin{aligned} \Delta_1 &= (1) \\ \Delta_2 &= (1)(1) - 1(1+1) = (1)(1) - (2) \\ \Delta_3 &= (1)[(1)(1) - (2)] - 2[(2)(1) - (3)] = (1)(1)(1) - (1)(2) \\ &\quad - 2(2)(1) + 2(3) \\ \Delta_4 &= (1)(1)(1)(1) - (1)(1)(2) - 2(1)(2)(1) + 2(1)(3) - 3(2)(1)(1) \\ &\quad + 3(2)(2) + 6(3)(1) - 6(4) \\ \Delta_5 &= (1)(1)(1)(1)(1) - (1)(1)(1)(2) - 2(1)(1)(2)(1) + 2(1)(1)(3) \\ &\quad - 3(1)(2)(1)(1) + 3(1)(2)(2) + 6(1)(3)(1) - 6(1)(4) \\ &\quad - 4(2)(1)(1)(1) + 4(2)(1)(2) + 8(2)(2)(1) - 8(2)(3) \\ &\quad + 12(3)(1)(1) - 12(3)(2) - 24(4)(1) + 24(5) \\ &\quad \text{etc.} \end{aligned} \right\} \{28\}$$

By collection of repeated terms and recalling that $(1^r) = \Delta_r$, the expansion becomes

$$\left. \begin{aligned} (1) &= (1) \\ (1^2) &= (1)^2 - (2) \\ (1^3) &= (1)^3 - 3(2)(1) + 2(3) \\ (1^4) &= (1)^4 - 6(2)(1)^2 + 3(2)^2 + 8(3)(1) - 6(4) \\ (1^5) &= (1)^5 - 10(2)(1)^3 + 15(2)(2)(1) + 20(3)(1)(1) - 20(3)(2) \\ &\quad - 30(4)(1) + 24(5). \\ &\dots\dots\dots \end{aligned} \right\} \{29\}$$

It is possible to write values of (1^r) in terms of power sums though the practical difficulty increases as r increases. Also continued use of the recursion formula {27} is apt to lead to error. Two simple checks are available. If D_r represents the sum of the coefficients of the expansion of (1^r) and $|D_r|$ represents the sum of the absolute values of these coefficients, then

$$D_r = 0 \quad \text{when } r > 1 \quad \{30\}$$

$$|D_r| = r! \quad \{31\}$$

The proof of {30} and {31} follows directly from {27} since the coefficients of Δ_r and $\mathbb{A}_r$ are the same. Thus $D_{r+1} = (1 - r)D_r$ and $|D_{r+1}| = (1 + r)|D_r|$. Since $D_2 = 0$ it follows that $D_3, D_4, \dots, D_r = 0$ and since $|D_2| = 2!$ it follows that $|D_3|, |D_4|, \dots, |D_r|$ are $3!, 4!, \dots, r!$ respectively.

26. Determination of the Coefficient of Any Ordered Product of Power Sums in the Expansion of Δ_r . We next attempt to revise the process outlined above so as to get the formulas {29} without going through the work of writing out {28}. We note first that every product of power sums in the expansion of (1^r) in {28} has been obtained from (1) by a succession of $r - 1$ operations which were either prefixes (when the (1) was prefixed) or raises (when the (1) was added). Also the order of the power sums in a given term indicates which operations have been prefixes and which raises. For example (1)(1)(1)(1)(1) results from 4 prefixes while (5) results from 4 raises. The term (3)(2) results from 1 raise, 1 prefix, and 2 raises respectively, while the term (2)(3) results from 2 raises, a prefix, and a raise. The product $(p_4)(p_3)(p_2)(p_1)$ results from prefixes when $r = p_1, r = p_1 + p_2, r = p_1 + p_2 + p_3$ and raises at all other times.

The sign of the coefficient of $(p_4)(p_3)(p_2)(p_1)$ can be determined when we recall that each raise is accompanied by a multiplication by $-r$ while each prefix is accompanied by no change in the coefficient. There have been $p_1 - 1 + p_2 - 1 + p_3 - 1 + p_4 - 1 = p_1 + p_2 + p_3 + p_4 - 4$ raises so the sign is $(-1)^{r-\rho}$ where $p_1 + p_2 + p_3 + p_4 = r$. More generally if $(p_\rho) \dots (p_3)(p_2)(p_1)$ is a term in the expansion of (1^r) where $p_\rho + \dots + p_3 + p_2 + p_1 = r$ the number of changes in the sign is $p_1 - 1 + p_2 - 1 + \dots + p_\rho - 1 = p_1 + p_2 + \dots + p_\rho - \rho = r - \rho$. It follows at once that those products of power sums in the expansion of (1^r) which have the same number of factors, ρ , also have the same sign and that this sign is $(-1)^{r-\rho}$.

In determining the numerical part of the coefficient we note that each prefix is accompanied by a multiplication by unity which can be written in the form $\frac{r}{r}$. Each raise is accompanied by a multiplication by r so there appears in the numerator the product of all possible values of r and in the denominator the product of those values of r corresponding to each prefix. For example the numerical coefficient of $(p_4)(p_3)(p_2)(p_1)$ is

$$\frac{(p_4 + p_3 + p_2 + p_1 - 1)!}{(p_3 + p_2 + p_1)(p_2 + p_1)(p_1)} = \frac{(p_4 + p_3 + p_2 + p_1)!}{(p_4 + p_3 + p_2 + p_1)(p_3 + p_2 + p_1)(p_2 + p_1)(p_1)}$$

Similarly the coefficient, without sign of $(p_\rho)(p_{\rho-1}), \dots (p_3)(p_2)(p_1)$ in the expansion of (1^r) is

$$\frac{(p_\rho + p_{\rho-1} + \dots + p_3 + p_2 + p_1)!}{(p_\rho + p_{\rho-1} + \dots + p_3 + p_2 + p_1)(p_{\rho-1} + \dots + p_3 + p_2 + p_1) \dots (p_3 + p_2 + p_1)(p_2 + p_1)(p_1)} \quad \{32\}$$

The denominator of {32} has a certain resemblance to a factorial. Thus $4! = (1 + 1 + 1 + 1)(1 + 1 + 1)(1 + 1)(1)$ in which the successive factors are found by dropping the first unit. The corresponding algebraic expression $(p_4 + p_3 + p_2 + p_1)(p_3 + p_2 + p_1)(p_2 + p_1)(p_1)$ is found in the same way and might be called an "algebraic factorial." It might be designated by

$$(p_4 + p_3 + p_2 + p_1)i$$

It should be noted that the order of the terms in the algebraic factorial is significant. Thus $(p_2 + p_1)i \neq (p_1 + p_2)i$ unless $p_1 = p_2$.

The coefficient of $(p_\rho)(p_{\rho-1}) \dots (p_2)(p_1)$ in the expansion of (1^r) may now be written

$$(-1)^{r-\rho} \frac{r!}{(p_\rho + \dots + p_2 + p_1)i} \quad \{33\}$$

For example the coefficient

$$(2)(1)(2) \text{ is } (-1)^{5-3} \frac{5!}{5 \cdot 3 \cdot 2} = 4$$

$$(1)(2)(2) \text{ is } (-1)^{5-3} \frac{5!}{5 \cdot 4 \cdot 2} = 3$$

$$(2)(2)(1) \text{ is } (-1)^{5-3} \frac{5!}{5 \cdot 3 \cdot 1} = 8$$

and the total coefficient of all terms involving $(2)(2)(1)$ is 15.

With a less formal notation we might designate the sum of the $\rho!$ "algebraic factorials" which can be formed from $p_\rho, p_{\rho-1}, \dots, p_2, p_1$ by

$$\sum (p_\rho + p_{\rho-1} + \dots + p_2 + p_1)i$$

and the sum of their reciprocals by

$$\sum \frac{1}{(p_\rho + p_{\rho-1} + \dots + p_2 + p_1)i}$$

This notation calls for the inclusion of all the $\rho!$ algebraic factorials even though some of them may be alike. If

$$\pi_1, \pi_2, \dots, \pi_s$$

indicate the numbers of repeated p 's

$$\begin{aligned} \sum \frac{1}{(p_\rho + p_{\rho-1} + \cdots + p_2 + p_1)_i} \\ = \pi_1! \pi_2! \cdots \pi_s! \sum' \frac{1}{(p_\rho + p_{\rho-1} + \cdots + p_2 + p_1)_i} \end{aligned} \quad \{34\}$$

where $\sum'$ holds for the $\frac{r!}{\pi_1! \pi_2! \cdots \pi_s!}$ non-repeated terms.

In general the total coefficient of $(p_1)(p_2) \cdots (p_\rho)$ in the expansion of (1') is obtained by adding all possible terms {33} in which the same p 's occur in different positions in the product. Every possible different position grouping of the p 's is present but once since it is dependent solely on the unique order in which prefixes and raises have been combined to produce that particular position grouping. The number of these position groupings varies with the number of repeated p 's. The sum of the coefficients of these position groupings of the same p 's, i.e. the total coefficient of $(p_1)(p_2) \cdots (p_\rho)$ is then given by

$$(-1)^{r-\rho} r! \sum' \frac{1}{(p_\rho + \cdots + p_1)_i}$$

which can be written by means of {34}

$$(-1)^{r-\rho} \frac{r!}{\pi_1! \pi_2! \cdots \pi_s!} \sum \frac{1}{(p_\rho + \cdots + p_1)_i}.$$

The formula for (1) may be written

$$(1') = \sum (-1)^{r-\rho} \frac{r!}{\pi_1! \pi_2! \cdots \pi_s!} \sum \frac{1}{(p_\rho + \cdots + p_1)_i} (p_1)(p_2) \cdots (p_\rho) \quad \{35\}$$

27. Theorem on Algebraic Factorials. The result {35} can be further simplified by the theorem

$$\sum \frac{1}{(p_\rho + \cdots + p_2 + p_1)_i} = \frac{1}{p_\rho p_{\rho-1} \cdots p_2 p_1} \quad \{36\}$$

which is proved by mathematical induction.

A. It is true when $\rho = 2$, since

$$\sum \frac{1}{(p_2 + p_1)_i} = \frac{1}{(p_2 + p_1)_i} + \frac{1}{(p_1 + p_2)_i} = \frac{1}{p_2 + p_1} \left[\frac{1}{p_1} + \frac{1}{p_2} \right] = \frac{1}{p_1 p_2}$$

B. If it is true for $\rho = k$, it is true for $\rho = k + 1$ since

$$\begin{aligned} \sum \frac{1}{(p_{k+1} + p_k + \cdots + p_2 + p_1)_i} \\ = \frac{1}{p_{k+1} + \cdots + p_2 + p_1} \left[\sum \frac{1}{(p_k + \cdots + p_1)_i} \right. \\ \left. + \sum_{p_{k+1}} \sum \frac{1}{(p_k + p_{k-1} + \cdots + p_1)_i} \right] \end{aligned}$$

where $\sum_{p_{k+1}}$ gives the k terms in which p_{k+1} replaces $p_k, p_{k-1}, \dots, p_2, p_1$ respectively. Now if {36} is true when $\rho = k$

$$\sum \frac{1}{(p_{k+1} + \dots + p_1)!} = \frac{1}{p_{k+1} + \dots + p_1} \left[\frac{p_{k+1} + p_k + p_{k-1} + \dots + p_2 + p_1}{p_{k+1} p_k p_{k-1} \dots p_2 p_1} \right] = \frac{1}{p_{k+1} p_k \dots p_2 p_1}.$$

C. Hence it is true when $k = 2, 3, 4 \dots$.

28. **Formulas for (1^r).** Formula {35} may now be written

$$(1^r) = \sum (-1)^{r-\rho} \frac{r!}{\pi_1! \dots \pi_s!} \frac{1}{p_1 p_2 \dots p_\rho} (p_1)(p_2) \dots (p_\rho) \quad \{37\}$$

or if the p 's are ordered it may be written as

$$(1^r) = \sum (-1)^{r-\rho} \frac{r!}{\pi_1! \dots \pi_s!} \frac{1}{p_1^{\pi_1} \dots p_s^{\pi_s}} (p_1)^{\pi_1} \dots (p_s)^{\pi_s} \quad \{38\}$$

which is the formula previously given as {23} and {26}. In addition the check formulas {30} and {31} become

$$\sum (-1)^{r-\rho} \frac{r!}{p_1^{\pi_1} \dots p_s^{\pi_s} \pi_1! \dots \pi_s!} = 0 \quad \{39\}$$

$$\sum \frac{r!}{p_1^{\pi_1} \dots p_s^{\pi_s} \pi_1! \dots \pi_s!} = r! \quad \{40\}$$

These relations {39} and {40} correspond to statements of Cauchy (2), (I; 1; 252-3) and to later remarks of Cayley (D; 577). By dividing by $r!$, they become

$$\sum \frac{(-1)^{r-\rho}}{p_1^{\pi_1} \dots p_s^{\pi_s} \pi_1! \dots \pi_s!} = 0 \quad \{41\}$$

$$\sum \frac{1}{p_1^{\pi_1} \dots p_s^{\pi_s} \pi_1! \dots \pi_s!} = 1 \quad \{42\}.$$

The formula {38} is easily applied. Thus

$$\begin{aligned} (1^5) &= \frac{5!}{5} (5) - \frac{5!}{4} (4)(1) - \frac{5!}{32} (3)(2) + \frac{5!}{32!} (3)(1)^2 + \frac{5}{222!} (2)^2(1) \\ &\quad - \frac{5!}{23!} (2)(1)^3 + \frac{5!}{5!} (1)^5 \\ &= 24(5) - 30(4)(1) - 20(3)(2) + 20(3)(1)^2 + 15(2)^2(1) \\ &\quad - 10(2)(1)^3 + (1)^5. \end{aligned}$$

with

$$24 - 30 - 20 + 20 + 15 - 10 + 1 = 0$$

$$24 + 30 + 20 + 20 + 15 + 10 + 1 = 5!$$

We next write the formula (1^r) in such form that we use the principle of generalization from symmetry. If we multiply numerator and denominator of {38} by $(p_1 - 1)!^{\pi_1}(p_2 - 1)!^{\pi_2} \dots (p_s - 1)!^{\pi_s}$ we get

$$(1^r) = \sum (-1)^{r-\rho} (p_1 - 1)!^{\pi_1} (p_2 - 1)!^{\pi_2} \dots (p_s - 1)!^{\pi_s} \frac{r! (p_1)^{\pi_1} \dots (p_s)^{\pi_s}}{(p_1!)^{\pi_1} (p_2!)^{\pi_2} \dots (p_s!)^{\pi_s} \pi_1! \dots \pi_s!}$$

which immediately becomes

$$(1^r) = \sum (-1)^{r-\rho} (p_1 - 1)!^{\pi_1} (p_2 - 1)!^{\pi_2} \dots (p_s - 1)!^{\pi_s} \binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}} (p_1)^{\pi_1} \dots (p_s)^{\pi_s} \quad \{43\}.$$

This somewhat formidable appearing formula is easy to apply. For example, in finding the value of (1^5) we write in one row all possible partitions of 5. In the next row we place the well known values of $\binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}}$. In the next row we place the indicated products with proper signs. Thus

1^5	21^3	2^21	31^2	32	41	5
1	10	15	10	10	5	1
1	-1	+1	+2	-2	-6	+24

results in

$$(1^5) = (1)^5 - 10(2)(1)^3 + 15(2)^2(1) + 20(3)(1)^2 - 20(3)(2) - 30(4)(1) + 24(5)$$

as indicated above.

It is immediately recognized that formula {43} can be obtained from formula {3} by placing $P(1^r) = (1^r); p_1^{\pi_1} \dots p_s^{\pi_s}$ by $(p_1)^{\pi_1} \dots (p_s)^{\pi_s}$ and $P_{p_1^{\pi_1} \dots p_s^{\pi_s}}$ by $(-1)^{r-\rho} (p_1 - 1)!^{\pi_1} \dots (p_s - 1)!^{\pi_s}$ and hence that formulas of Table I may be used in obtaining the values of (1^r) .

29. Values of $(a_1 \dots a_r)$. The form of {43} also permits generalization from symmetry since the $\binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}}$ equal values $(p_1)^{\pi_1} \dots (p_s)^{\pi_s}$ are replaced by the $\binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}}$ different values composing $T(p_1)^{\pi_1} \dots (p_s)^{\pi_s}$ when the r units are replaced by the a 's. It follows at once that

$$(a_1 a_2 \dots a_r) = \sum (-1)^{r-\rho} (p_1 - 1)!^{\pi_1} \dots (p_s - 1)!^{\pi_s} T(p_1)^{\pi_1} \dots (p_s)^{\pi_s} \quad \{44\}$$

where

$$r = p_1 \pi_1 + p_2 \pi_2 + \dots + p_s \pi_s$$

and

$$\rho = \pi_1 + \pi_2 + \dots + \pi_s.$$

As an illustration we write

$$(abc) = 2(a + b + c) - (a + b)(c) - (a + c)(b) - (b + c)(a) \\ + (a)(b)(c)$$

as indicated earlier by {18} and

$$(a_1a_2a_3a_4) = -6T(4) + 2T(3)(1) + T(2)(2) - T(2)(1)^2 + T(1)^4 \\ (a_1a_2a_3a_4a_5) = 24T(5) - 6T(4)(1) - 2T(3)(2) + 2T(3)(1)^2 \\ + T(2)^2(1) - T(2)(1)^3 + T(1)^5 \\ \text{etc.}$$

30. Table of Values of $(a_1 \dots a_r)$. The values of the power products with $w \leq 6$ are given in Table II which follows the general form of Table I. In fact Table II may be derived from Table I by placing every

$$P_{p_1^{\pi_1} \dots p_s^{\pi_s}} = (-1)^{r-\rho} (p_1 - 1)!^{\pi_1} \dots (p_s - 1)!^{\pi_s}$$

as indicated in the next section.

31. Use of Partition Formulas. By comparing {44} with {4} we see that {44} can be obtained from {4} by placing

$$P(a_1a_2 \dots a_s) = (a_1a_2 \dots a_r) \\ P_{p_1^{\pi_1} \dots p_s^{\pi_s}} = (-1)^{r-\rho} (p_1 - 1)!^{\pi_1} \dots (p_s - 1)!^{\pi_s}$$

and $T_{p_1^{\pi_1} \dots p_s^{\pi_s}} = T(p_1)^{\pi_1} \dots (p_s)^{\pi_s}$

It appears then that the values of any power product sum $(a_1a_2 \dots a_r)$ can be obtained by writing the expansion of $P(a_1 \dots a_r)$ and substituting as indicated. Thus since

$$P(321) = P_36 + P_{21}51 + P_{21}42 + P_{21}33 + P_{111}111 \\ (321) = 2(6) - (5)(1) - (4)(2) - (3)(3) + (1)(1)(1).$$

It is also immediately apparent that Table II can be obtained from Table I by placing $P_{p_1^{\pi_1} \dots p_s^{\pi_s}}$ equal to $(-1)^{r-\rho} (p_1 - 1)!^{\pi_1} \dots (p_s - 1)!^{\pi_s}$ and that the main results of Chapter I, including the recursion rule, are applicable to the present problem.

32. Coefficients of Given Terms in the Expansion of Product Power Sums. The methods of the last section are also useful in finding the coefficient of any term in the expansion. For example we wish to find the coefficient of (3)(2) in the expansion of (2111). We note that $P\left(\begin{smallmatrix} 2111 \\ 32 \end{smallmatrix}\right) = P_{31} + 3P_{22}$ and that

TABLE II

Power product sums in terms of products of power sums when $W \leq 6$
 $W = 6$

	6	51	42	33	411	321	222	31 ³	2 ² 1 ²	21 ⁴	1 ⁶
6	1										
51	-1	1									
42	-1		1								
33	-1			1							
411	2	-2	-1		1						
321	2	-1	-1	-1		1					
222	2		-3				1				
31 ³	-6	6	3	2	-3	-3		1			
2 ² 1 ²	-6	4	5	2	-1	-4	-1		1		
21 ⁴	24	-24	-18	-8	12	20	3	-4	-6	1	
1 ⁶	-120	144	90	40	-90	-120	-15	40	45	-15	1

 $W = 1$

	1
1	1

 $W = 2$

	2	11
2	1	
11	-1	1

 $W = 3$

	3	21	111
3	1		
21	-1	1	
111	2	-3	1

 $W = 5$

	5	41	32	311	221	2111	11111
5	1						
41	-1	1					
32	-1		1				
311	2	-2	-1	1			
221	2	-1	-2		1		
2111	-6	6	5	-3	-3	1	
11111	24	-30	-20	20	+15	-10	1

 $W = 4$

	4	31	22	211	1111
4	1				
31	-1	1			
22	-1		1		
211	2	-2	-1	1	
1111	-6	8	3	-6	1

the coefficient of (3)(2) is $P_{31} + 3P_{22}$ where $P_{31} = (-1)^{4-2}2! = 2$ and $P_{22} = (-1)^{4-2} = 1$. Hence the coefficient is $1 \cdot 2 + 3 \cdot 1 = 5$.

33. The Expansion of the Monomial Symmetric Function. If $a_1 \asymp a_2 \asymp a_3 \asymp \dots \asymp a_r$ then $M(a_1 \dots a_r) = (a_1 \dots a_r)$ and previous results are applicable. If however the product power sum is of the form

$$(a_1^{\alpha_1} a_2^{\alpha_2} \dots a_h^{\alpha_h})$$

then

$$M(a_1^{\alpha_1} a_2^{\alpha_2} \dots a_h^{\alpha_h}) = \frac{1}{\alpha_1! \alpha_2! \dots \alpha_h!} (a_1^{\alpha_1} \dots a_h^{\alpha_h})$$

and

$$M(a_1^{\alpha_1} \dots a_h^{\alpha_h}) = \frac{1}{\alpha_1! \alpha_2! \dots \alpha_h!} \sum (-1)^{r-\rho} (p_1 - 1)!^{\pi_1} \dots (p_s - 1)!^{\pi_s} T(p_1)^{\pi_1} \dots (p_s)^{\pi_s} \quad \{45\}$$

For example

$$M(421) = 2(7) - (6)(1) - (5)(2) - (4)(3) + (4)(2)(1)$$

$$M(322) = (7) - (5)(2) - \frac{1}{2}(4)(3) + \frac{1}{2}(3)(2)(2).$$

$$M(2^2 1^2) = -\frac{3}{2}(6) + (5)(1) + \frac{5}{4}(4)(2) + \frac{1}{2}(3)(3) - \frac{1}{4}(4)(1)(1) \\ - (3)(2)(1) - \frac{1}{4}(2)(2)(2) + \frac{1}{4}(2)(2)(1)(1).$$

Study will show that the formula {45} is equivalent to one given by Faà de Bruno (C; 9) and later by Roe (7).

It is possible to use Table II in finding the expansion of the monomial symmetric functions. It is only necessary to multiply each term in the expansion

of $(a_1 \dots a_r)$ by $\frac{1}{\alpha_1! \dots \alpha_h!}$.

The check formulas give, in the case of the monomial symmetric function:

The sum of the coefficients in the expansion is 0.

The sum of the absolute values of the coefficients is $\frac{r!}{\alpha_1! \alpha_2! \dots \alpha_h!}$.

The reader might compare the second of these checks with the results of Faà de Bruno (C; 14).

Tables giving the expansion of monomial symmetric function have been given. One by J. R. Roe (12; plate 18) includes all cases of weight ≤ 10 .

34. Previous Results. Previous authors have studied the monomial symmetric function. Gordan has deduced a monomial symmetric function formula which is recommended by J. R. Roe (M; 24-33). MacMahon has given a general formula (K; II; 320) for expanding any monomial symmetric function in terms of power sums together with an operational method for its evaluation. O'Toole also has given a differential operator and showed how it could be applied in obtaining expansions (16; 115-130). O'Toole has also given a method of expanding symmetric functions in many variables by means of differential operators, (17).

Another method of attack was based upon the close relation existing between the elementary symmetric function and the determinant of the power sums. This has resulted in the expression of the monomial symmetric function in determinant form. Brioschi appears to have been the first (1854) to see how

a symbolic determinant could be used (3; 427) although he gave no proof. Bellavites tried in 1857, but obtained incorrect results (4). In 1876 Faà de Bruno made an attempt, but he too was in error (C; 10). In 1898 E. D. Roe, Jr. proved that Brioschi was right (7). Muir also gave a proof in 1908 (11; 5-9). The summation of determinants, rather than the symbolic determinant, was used by Hankel (6; 90-94) (L; III, 220).

The determinant of the power sums has been generalized in another way. A group of writers has studied the "immanents" of its matrix. D. E. Littlewood and A. R. Richardson have recently written a series of papers on this topic. One of these papers (18; 99-141) defined the term "immanents" and gave references to previous investigations dealing with this matrix.

It has been the aim of this chapter to present an easy development of the subject of the expansion of product power sums and monomial symmetric functions. This development is characterized by

1. The use of the formulas and tables of Chapter I in writing expansions of product power sums.
2. The use of product power sums in place of monomial symmetric functions which makes feasible
3. Generalization from symmetry.
4. References to previous work.

Chapter IV. The Double Expansion Theorem

In the present chapter we combine the multiplication expansion of Chapter II and the power product sum expansion of Chapter III into a new result which is to be known as the double expansion theorem. We show that this result may also be expressed in terms of the partition notation of Chapter I.

35. **The Value of $K(a_1)(a_2)$.** We know

$$(a_1)(a_2) = (a_1 + a_2) + (a_1 \cdot a_2)$$

and if we multiply $(a_1 + a_2)$ by k_2 and $(a_1 \cdot a_2)$ by k_{11} we have a new expression which we designate by $K(a_1)(a_2)$.

$$K(a_1)(a_2) = k_2(a_1 + a_2) + k_{11}(a_1 \cdot a_2) \quad \{46\}$$

Since $(a_1 a_2) = (a_1)(a_2) - (a_1 + a_2)$

$$K(a_1)(a_2) = k_2(a_1 + a_2) + k_{11}[(a_1)(a_2) - (a_1 + a_2)]$$

$$K(a_1)(a_2) = (k_2 - k_{11})(a_1 + a_2) + k_{11}(a_1)(a_2)$$

which can be written

$$K(a_1)(a_2) = K_2(a_1 + a_2) + K_{11}(a_1)(a_2) \quad \{47\}$$

if $k_2 - k_{11} = K_2$ and $K_{11} = k_{11}$

36. **The Value of $K(a_1)(a_2)(a_3)$.** We know from {12} that

$$(a_1)(a_2)(a_3) = T(3) + T(21) + T(111)$$

and we define $K(a_1)(a_2)(a_3) = k_3T(3) + k_{21}T(21) + k_{111}T(111)$. Inserting the values $T_3 = (a_1 + a_2 + a_3)$, $T_{21} = (a_1 + a_2 \cdot a_3) + (a_1 + a_3 \cdot a_2) + (a_2 + a_3 \cdot a_1)$, $T_{111} = (a_1a_2a_3)$ and reducing to power sums by {44}, we get

$$K(a_1)(a_2)(a_3) = (k_3 - 3k_{21} + 2k_{111})(a_1 + a_2 + a_3) + (k_{21} - k_{111}) \\ \{(a_1 + a_2)(a_3) + (a_1 + a_3)(a_2) + (a_2 + a_3)(a_1)\} + k_{111}(a_1)(a_2)(a_3)$$

which may be written

$$K(a_1)(a_2)(a_3) = K_3(a_1 + a_2 + a_3) + K_{21}\{(a_1 + a_2)(a_3) \\ + (a_1 + a_3)(a_2) + (a_2 + a_3)(a_1)\} + K_{111}(a_1)(a_2)(a_3) \quad \{48\}$$

where $K_3 = k_3 - 3k_{21} + 2k_{111}$, $K_{21} = k_{21} - k_{111}$, $K_{111} = k_{111}$.

37. **Definition of $K(a_1)(a_2) \dots (a_r)$.** We define

$$K(a_1)(a_2) \dots (a_r) = \sum k_{p_1^{\pi_1} \dots p_s^{\pi_s}} T(p_1^{\pi_1} \dots p_s^{\pi_s}) \quad \{49\}$$

where $T(p_1^{\pi_1} \dots p_s^{\pi_s})$ is composed of $\binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}}$ power product sums. We wish to find the value $K(a_1)(a_2) \dots (a_r)$ in terms of power sums. This involves the expansion of each power product sum in terms of power sums and then the collection of the results. This algebraic process is to be called the double expansion process and the theorem which results, the double expansion theorem.

38. **Special Cases of the Theorem.** The results {47} and {48} are special cases of the double expansion theorem when $r = 2, 3$. When $r = 1$ it is evident that $K(a_1) = K_1(a_1) = k_1(a_1)$. {50}

The results {50}, {48}, and {49} may be written symbolically by

$$K(a_1) = K_1T(1) \\ K(a_1)(a_2) = K_2T(2) + K_{11}T(1)^2 \\ K(a_1)(a_2)(a_3) = K_3T(3) + K_{21}T(2)(1) + K_{111}T(1)^3$$

It can also be shown, with a much more extensive use of the results of Chapters II and III, that

$$K(a_1)(a_2)(a_3)(a_4) = K_4T(4) + K_{31}T(3)(1) + K_{22}T(2)^2 \\ + K_{211}T(2)(1)^2 + K_{1111}T(1)^4 \quad \{52\}$$

$$K(a_1) \dots (a_5) = K_5T(5) + K_{41}T(4)(1) + K_{32}T(3)(2) \\ + K_{311}T(3)(1)^2 + K_{221}T(2)^2(1) + K_{2111}T(2)(1)^3 \\ + K_{11111}T(1)^5 \quad \{53\}$$

where

$$\left. \begin{aligned} K_4 &= k_4 - 4k_{31} - 3k_{22} + 12k_{211} - 6k_{1111} \\ K_{31} &= k_{31} - 3k_{211} + 2k_{1111} \\ K_{22} &= k_{22} - 2k_{211} + k_{1111} \\ K_{211} &= k_{211} - k_{1111} \\ K_{1111} &= k_{1111} \end{aligned} \right\} \{54\}$$

and

$$\left. \begin{aligned} K_5 &= k_5 - 5k_{41} - 10k_{32} + 20k_{311} + 30k_{221} - 60k_{2111} + 24k_{11111} \\ K_{41} &= k_{41} - 4k_{311} - 3k_{221} + 12k_{2111} - 6k_{11111} \\ K_{32} &= k_{32} - k_{311} - 3k_{221} + 5k_{2111} - 2k_{11111} \\ K_{311} &= k_{311} - 3k_{2111} + 2k_{11111} \\ K_{221} &= k_{221} - 2k_{2111} + k_{11111} \\ K_{2111} &= k_{2111} - k_{11111} \\ K_{11111} &= k_{11111} \end{aligned} \right\} \{55\}$$

We may say then that, for $r < 6$

$$\begin{aligned} K(a_1) \cdots (a_r) &= \sum k_{p_1^{\pi_1} \dots p_s^{\pi_s}} T(p_1^{\pi_1} \dots p_s^{\pi_s}) \\ &= \sum k_{p_1^{\pi_1} \dots p_s^{\pi_s}} T(p_1)^{\pi_1} \dots (p_s)^{\pi_s} \end{aligned} \quad \{56\}$$

where $K_{p_1^{\pi_1} \dots p_s^{\pi_s}}$ is defined by the relations {47}, {48}, {54}, and {55}. In examining the value of K_r we note

$$\begin{aligned} K_1 &= k_1 \\ K_2 &= k_2 - k_{11} \\ K_3 &= k_3 - 3k_{21} + 2k_{111} \\ K_4 &= k_4 - 4k_{31} - 3k_{22} + 12k_{211} - 6k_{1111} \\ K_5 &= k_5 - 5k_{41} - 10k_{32} + 20k_{311} + 30k_{221} - 60k_{2111} + 24k_{11111} \end{aligned}$$

and that these are given, for $r < 6$ by

$$K_r = \sum (-1)^\rho (\rho - 1)! \binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}} k_{p_1^{\pi_1} \dots p_s^{\pi_s}} \quad \{57\}$$

It is further to be noted that {57} can be obtained from {3} by placing $P(1^r) = K_r$, $p_1^{\pi_1} \dots p_s^{\pi_s}$ by $k_{p_1^{\pi_1} \dots p_s^{\pi_s}}$, and $P_{p_1^{\pi_1} \dots p_s^{\pi_s}}$ by $(-1)^\rho (\rho - 1)!$. Hence the last rows of Table I may be used in writing the values of K_r . Thus from

$$P(1^3) = P_3(3) + 3P_{21}(21) + P_{111}(1)^3$$

we get

$$K_3 = k_3 - 3k_{21} + 2k_{111}.$$

It is further evident that if $\overline{K_3 K_2} = \overline{(k_3 - 3k_{21} + 2k_{111})(k_2 - k_{11})}$ indicates multiplication by suffixing of subscripts that $\overline{K_3 K_2} = k_{32} - k_{311} - 3k_{221} + 5k_{211} - 2k_{1111} = K_{32}$

and in general it can be shown that for $r < 6$

$$\begin{aligned} K_{r_1 r_2} &= \overline{K_{r_1} K_{r_2}} \\ K_{r_1 r_2 r_3} &= \overline{K_{r_1} K_{r_2} K_{r_3}} \\ &\text{etc.} \end{aligned} \quad \{58\}$$

so that all values $K_{p_1^{x_1} \dots p_s^{x_s}}$ may be obtained by symbolic multiplication of equations {57}.

The method of this section can be used in demonstrating that the results {56}, {57}, and {58} hold also when $r = 6, 7, 8 \dots$, but the amount of algebraic manipulation increases enormously with each increase in r . We establish these results, for all integral values of r , by a more general approach.

39. A More General Definition. We provide a more general definition of $K(a_1) \dots (a_r)$ by letting the subscripts of the k 's agree with parts of the given partition rather than with its complete order. Thus

$$K'(a_1)(a_2) = k_{a_1+a_2}(a_1 + a_2) + k_{a_1 a_2}(a_1 a_2)$$

and in general, if $q_1^{x_1} \dots q_t^{x_t}$ represents any ρ part partition having complete order $p_1^{x_1} \dots p_s^{x_s}$, then we may define

$$K'(a_1)(a_2) \dots (a_r) = \sum k_{q_1^{x_1} \dots q_t^{x_t}}(q_1^{x_1} \dots q_t^{x_t}) \quad \{59\}$$

where the summation holds, not only for every different complete order as does {49}, but for every possible partition. By {44} $(q_1^{x_1} \dots q_t^{x_t})$ may be written as

$$(q_1^{x_1} \dots q_t^{x_t}) = \sum (-1)^{\rho-g} (d_1 - 1)! (d_2 - 1)! \dots (d_g - 1)! T(d_1) \dots (d_g)$$

where $d_1 + d_2 + \dots + d_g = \rho$ and where groups of the d 's may be alike. If $(w_1)(w_2) \dots (w_g)$ is one of the products of power sums having the complete order $(d_1 \dots d_g)$ we may write

$$(q_1^{x_1} \dots q_t^{x_t}) = \sum (-1)^{\rho-g} (d_1 - 1)! \dots (d_g - 1)! (w_1)(w_2) \dots (w_g) \quad \{60\}$$

where

$$q_1 x_1 + \dots + q_t x_t = w = w_1 + w_2 + \dots + w_g$$

and

$$x_1 + \dots + x_t = \rho$$

and where the summation sign holds not only for every complete order $d_1 \cdots d_g$, but for all power sum partition products $(w_1)(w_2) \cdots (w_g)$.

The insertion of {60} in {59} gives

$$K'(a_1)(a_2) \cdots (a_r) = \sum k_{q_1^{x_1} \cdots q_g^{x_g}} \sum (-1)^{\rho-g} (d_1 - 1)! \cdots (d_g - 1)! (w_1) \cdots (w_g) \quad \{61\}$$

40. **Value of K'_w .** The notation of K'_w is used to indicate the coefficient of the power sum $(w) = (a_1 + a_2 + \cdots + a_r)$ in the expansion of {61}. In this case $d_1 = \rho$ and $g = 1$ so that

$$K'_w = \sum k_{q_1^{x_1} \cdots q_g^{x_g}} (-1)^{\rho-1} (\rho - 1)! \quad \{62\}$$

which may be written more symbolically as

$$K'_w = \sum (-1)^{\rho-1} (\rho - 1)! k_{\pi_w} \quad \{63\}$$

where π_w represents any algebraic partition of $a_1 + \cdots + a_r$ and ρ indicates the number of its parts.

41. **Products of K'' 's.** The notation $\overline{K'_{w_1} K'_{w_2}}$ is used to indicate the product of K'_{w_1} by K'_{w_2} if the rule of multiplication is the suffixing of the subscripts of the k 's in the expansion of K'_{w_1} and K'_{w_2} . Thus

$$\begin{aligned} \overline{K'_{a_1+a_2} \cdot K'_{a_3}} &= \overline{(k_{a_1+a_2} - k_{a_1 a_2})(k_{a_3})} \\ &= k_{a_1+a_2 \cdot a_3} - k_{a_1 a_2 a_3} \end{aligned}$$

More generally, if we write, from {63}

$$\begin{aligned} K'_{w_1} &= \sum (-1)^{d_1-1} (d_1 - 1)! k_{\pi_{w_1}} \\ K'_{w_2} &= \sum (-1)^{d_2-g} (d_2 - 1)! k_{\pi_{w_2}} \\ &\dots\dots\dots \\ K'_{w_g} &= \sum (-1)^{d_g-1} (d_g - 1)! k_{\pi_{w_g}} \end{aligned}$$

and use multiplication by suffixing of subscripts we have

$$\overline{K'_{w_1} K'_{w_2} \cdots K'_{w_g}} = \sum (-1)^{\rho-g} (d_1 - 1)! \cdots (d_g - 1)! k_{\pi_{w_1} \cdots \pi_{w_g}} \quad \{64\}$$

where $\rho = d_1 + d_2 + \cdots + d_g$ and the summation holds for every partition which can be formed by combining any algebraic partition of w_1 , any partition of w_2 , $\dots$, any partition of w_g .

42. **The Coefficient of $(w_1)(w_2) \cdots (w_g)$.** The coefficients of any specific product of power sums $(w_1)(w_2) \cdots (w_g)$ is from {61}

$$K'_{w_1 w_2 \cdots w_g} = \sum k_{q_1^{x_1} \cdots q_g^{x_g}} (-1)^{\rho-g} (d_1 - 1)! \cdots (d_g - 1)! \quad \{65\}$$

where the summation holds, not only for the partitions of $a_1 + a_2 + \dots + a_r$, but for the partitions $\pi_{w_1}, \pi_{w_2}, \dots, \pi_{w_g}$ since these partitions can be combined to form $(w_1)(w_2) \dots (w_g)$. Hence {65} becomes

$$K'_{w_1 w_2 \dots w_g} = \sum (-1)^{\rho-g} (d_1 - 1)! \dots (d_g - 1)! k_{\pi_{w_1} \dots \pi_{w_g}} \quad \{66\}$$

and it is immediately seen that the right hand expressions of {66} and {64} are the same and hence that

$$K'_{w_1 w_2 \dots w_g} = \overline{K'_{w_1} K'_{w_2} \dots K'_{w_g}}$$

as expected from (58).

We can now say that

$$\begin{aligned} K'(a_1)(a_2) \dots (a_r) &= \sum k_{q_1^{z_1} \dots q_t^{z_t}} (q_1^{z_1} \dots q_t^{z_t}) \\ &= \sum K'_{w_1 w_2 \dots w_r} (w_1)(w_2) \dots (w_g) \end{aligned} \quad \{67\}$$

where

$$K'_w = \sum (-1)^{\rho-1} (\rho - 1)! k_{\pi_w} \quad \{68\}$$

and

$$K'_{w_1 w_2 \dots w_g} = \overline{K'_{w_1} K'_{w_2} \dots K'_{w_g}} \quad \{69\}$$

Relations {67}, {68} and {69} constitute the general double expansion theorem.

43. The Double Expansion Theorem. The case of the double expansion theorem in which we are especially interested is that in which the coefficients of all similar power sum products are the same, i.e., $k_{q_1^{z_1} \dots q_s^{z_s}}$ is a function of the complete order indicated by $k_{p_1^{\pi_1} \dots p_s^{\pi_s}}$. In this case {68} becomes

$$K_w = \sum (-1)^\rho (\rho - 1)! \binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}} k_{p_1^{\pi_1} \dots p_s^{\pi_s}} \quad \{70\}$$

where the summation holds for all possible complete orders. Suppose now that the r algebraic expressions, $a_1, a_2, \dots, a_r$ are all unity then {69} becomes

$$K_r = \sum (-1)^{\rho-1} (\rho - 1)! \binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}} k_{p_1^{\pi_1} \dots p_s^{\pi_s}}$$

and we find that $K_w = K_r$. We may then write {67}, {68} and {69} as

$$K(a_1) \dots (a_r) = \sum k_{p_1^{\pi_1} \dots p_s^{\pi_s}} (p_1^{\pi_1} \dots p_s^{\pi_s}) = \sum K_{r_1 \dots r_g} T(r_1) \dots (r_g) \quad \{71\}$$

where

$$K_r = \sum (-1)^{\rho-1} (\rho - 1)! \binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}} k_{p_1^{\pi_1} \dots p_s^{\pi_s}} \quad \{72\}$$

and

$$K_{r_1 r_2 \dots r_g} = \overline{K_{r_1} K_{r_2} \dots K_{r_g}} \quad \{73\}$$

Now $r_1 r_2 \dots r_g$ indicates any grouping of the a 's, and hence any complete order of $a_1 + a_2 + \dots + a_r$. So {71} may be written, with a slight change of notation as

$$K(a_1) \dots (a_r) = \sum k_{p_1^{\pi_1} \dots p_s^{\pi_s}} T(p_1^{\pi_1} \dots p_s^{\pi_s}) \\ = \sum K_{p_1^{\pi_1} \dots p_s^{\pi_s}} T(p_1)^{\pi_1} \dots (p_s)^{\pi_s} \quad \{74\}$$

The relations {74}, {72} and {73} are the desired generalizations of {56}, {57} and {58} and hold for all positive integral values of r .

The double expansion theorem provides a method of writing out the result of the double expansion process without going through the work involved in the process. Thus

$$K(3)(2)(1) = K_3(6) + K_{21}\{(5)(1) + (4)(2) + (3)(3)\} \\ + K_{111}(3)(2)(1) = (k_3 - 3k_{21} + 2k_{111})(6) \quad \{75\} \\ + (k_{21} - k_{111})\{(5)(1) + (4)(2) + (3)(3)\} + k_{111}(3)(2)(1)$$

44. The Double Expansion Theorem and Partition Notation. It is immediately evident that {74} can be obtained from {4} if $P(a_1 \dots a_r)$ is replaced by $K(a_1) \dots (a_r)$, if $P_{p_1^{\pi_1} \dots p_s^{\pi_s}}$ is replaced by $K_{p_1^{\pi_1} \dots p_s^{\pi_s}}$, and if $p_1^{\pi_1} \dots p_s^{\pi_s}$ is replaced by $T(p_1)^{\pi_1} \dots (p_s)^{\pi_s}$. It follows at once that the entire theory of Chapter I,—table, recursion formula, etc.—is applicable to double expansion theory. For example {75} above is obtained from

$$P(321) = P_3 6 + P_{21}\{51 + 42 + 33\} + P_{111} 321$$

simply by replacing the K 's by the P 's and enclosing the parts in parentheses. We can as well use P 's as K 's to represent the double expansion theorem and hence have available a list of double expansion formulas when $w \leq 6$. We also have available a recursion property for writing double expansions beyond the scope of the table. Thus for example, the illustration at the end of section 9 may be interpreted as a statement of the double expansion theorem when $a_1 = 3, a_2 = 2, a_3 = 2, a_4 = 1$.

45. The Case of Equal Powers. In case $a_1 = a_2 = a_3 = \dots = a$, {74} reduces to {3'} of Chapter I with

$$P_r = \sum (-1)^{\rho-1} (\rho-1)! \binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}} k_{p_1^{\pi_1} \dots p_s^{\pi_s}} \quad \{76\}$$

and

$$P_{r_1 r_2 \dots r_g} = \overline{P_{r_1} P_{r_2} \dots P_{r_g}}.$$

Formula {74} also reduces to {3} when $a_1 = a_2 = \dots = a_r = 1$.

46. Special Values of $K_{p_1^{\pi_1} \dots p_s^{\pi_s}}$.

A. $k_{p_1^{\pi_1} \dots p_s^{\pi_s}} = 1$. In this case the coefficients are all unity and

$$P(a_1)(a_2) \dots (a_r) = (a_1)(a_2) \dots (a_r)$$

It follows that $P_r = 0$ and that $P_{p_1^{\pi_1} \dots p_s^{\pi_s}} = 0$ except that $P_r = 1$. Placing $P_r = 0$ and $k_{p_1^{\pi_1} \dots p_s^{\pi_s}} = 1$ in {72} or its equivalent {76} we have, when $r > 1$

$$0 = \sum (-1)^{\rho-1} (\rho - 1)! \binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}} \quad \{77\}$$

where the summation holds for every partition of r . This formula should be compared with {39} and {40}. When $r = 4$ and the partitions are

	4,	31,	22,	211,	1 ⁴	
{77} gives	1	-4	-3	+12	-6	= 0
{39} gives	-6	+8	+3	-6	+1	= 0
{40} gives	6	+8	+3	+6	+1	= 4!

The equivalent of {77} was first given by Cayley (D; 576) who at the same time noted the similarity to {39}.

It follows immediately that the sum of the coefficients in the expansion of $P_{p_1^{\pi_1} \dots p_s^{\pi_s}}$, except P_{1^r} , is 0, for the sum of the coefficients of $P_{p_1^{\pi_1} \dots p_s^{\pi_s}}$ is the sum of the coefficients of $(P_{p_1})^{\pi_1} \dots (P_{p_s})^{\pi_s}$ and is 0. For example the sum of the coefficient of $P_{32} = k_{32} - k_{311} - 3k_{221} + 5k_{2111} - 2k_{11111}$ is 0.

Since the coefficients of $(\mu_1')^{\pi_1} \dots (\mu_s')^{\pi_s}$ (19; 25) in the expansion of Thiele half invariants are $(-1)^{\rho-1} (\rho - 1)! \binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}}$ it follows from {77} that the sum of these coefficients is 0.

B. $k_{p_1^{\pi_1} \dots p_s^{\pi_s}} = \frac{n^{(\rho)}}{N^{(\rho)}}$. In this case all terms having the same number of parts, ρ , have the same coefficients. If we indicate $\frac{n^{(\rho)}}{N^{(\rho)}}$ by $\rho_1, \rho_2, \dots$, when $\rho = 1, 2, \dots$, {57}, {76} become

$$\begin{aligned} P_1 &= \rho_1 \\ P_2 &= \rho_1 - \rho_2 \\ P_3 &= \rho_1 - 3\rho_2 + 2\rho_3 \\ P_4 &= \rho_1 - 7\rho_2 + 12\rho_3 - 6\rho_4 \\ P_5 &= \rho_1 - 15\rho_2 + 50\rho_3 - 60\rho_4 + 24\rho_5. \\ &\text{etc.} \end{aligned}$$

which are the formulas which have been used by Carver (15) and O'Toole (16).

Many other additional cases can be obtained by giving different values to $k_{p_1^{\pi_1} \dots p_s^{\pi_s}}$, but a discussion of these is hardly justified here as the case in which $k_{p_1^{\pi_1} \dots p_s^{\pi_s}}$ is a function of the number of parts, ρ , is to be used in Part II.

47. Relation to Previous Results. No general statement of the double expansion theorem has previously been given although the special case $K(a')$

has been developed by Carver (15) and O'Toole (16). Their results are further restricted to the special case (B) of section 46. The application of the double expansion theorem in this case is very useful in studying sampling from a finite universe as Carver has shown and as is demonstrated in Part II.

Most writers who have worked on the problem of moments of moments have gone through the double expansion process, but Carver was the first to note that the result of the process can be written in terms of the P polynomials above. It seems appropriate therefore to refer to these P polynomials of the coefficients as Carver polynomials.

Chapter V. The Multipartition and Multivariate Formulas

It is the purpose of this chapter to show how the results of Chapters I, II, III, and IV may be extended to the case of different variables.

48. Multipartitions. Tables. Formula {4} is still applicable if we let the a_1 units be the units of one quantity, the a_2 units to be the units of a second quantity, etc. Thus for example the formula $P(a_1 a_2 a_3)$ may be used to represent the precise number of ways in which a_1 apples, a_2 pears and a_3 peaches can be formed into groups without breaking up the groups of apples, pears, and peaches.

Various conventions for representing multipartitions of this type have been used. We adopt the one in which the individual partitions are written in successive columns. The partitions of the first number are combined with the partitions of the second number to form all possible multipartitions. Thus the multipartite number 111 has the partitions

111	110	101	011	100
	001	010	100	010
				001

where the parts are given in the rows. It is desired to show the number of ways in which any one of those partitions may be combined to form partitions of fewer parts. Thus

$$P \begin{pmatrix} 100 \\ 010 \\ 001 \end{pmatrix} = P_3 111 + P_{21} 110 + P_{21} 101 + P_{21} 011 + P_{111} 100 + P_{111} 010 + P_{111} 001$$

This is obtained from $P(a_1 a_2 a_3)$ by placing $a_1 = 1_1$, $a_2 = 1_2$, $a_3 = 1_3$, and could be written from {4} as

$$P(1_1 1_2 1_3) = P_3(1_1 + 1_2 + 1_3) + P_{21}\{\overline{1_1 + 1_2} \cdot 1_3 + \overline{1_1 + 1_3} \cdot 1_2 + \overline{1_2 + 1_3} \cdot 1_1\} + P_{111} 1_1 \cdot 1_2 \cdot 1_3.$$

Similarly

$$P \begin{pmatrix} 10 \\ 10 \\ 01 \\ 01 \end{pmatrix} = P_4 \begin{matrix} 2 \\ 2 \end{matrix} + 2P_{31} \begin{matrix} 21 \\ 01 \end{matrix} + 2P_{31} \begin{matrix} 12 \\ 10 \end{matrix} + P_{22} \begin{matrix} 20 \\ 02 \end{matrix} + 2P_{11} \begin{matrix} 11 \\ 11 \end{matrix} + P_{211} \begin{matrix} 20 \\ 01 \end{matrix} + P_{211} \begin{matrix} 02 \\ 10 \end{matrix} \\ + 4P_{211} \begin{matrix} 11 \\ 01 \end{matrix} + P_{1111} \begin{matrix} 10 \\ 01 \end{matrix} + P_{1111} \begin{matrix} 10 \\ 01 \end{matrix}$$

is a special case of $P(a_1a_2a_3a_4)$ where $a_1 = 1_1, a_2 = 1_1, a_3 = 1_2, a_4 = 1_2$. Formula {4} is also true where the a_1 units are not of the same kind. Thus

$$P(a_1a_2) = P_2(a_1 + a_2) + P_{11}(a_1a_2)$$

gives

$$P \begin{pmatrix} 11 \\ 10 \end{pmatrix} = P_2 21 + P_{11} \begin{matrix} 11 \\ 10 \end{matrix} \quad \text{when } a_1 = 1_1 + 1_2 \text{ and } a_2 = 1_1.$$

TABLE III
The Multipartite Number 11

	11	$\begin{matrix} 10 \\ 01 \end{matrix}$
11	P_1	
$\begin{matrix} 10 \\ 01 \end{matrix}$	P_2	P_{11}

The Multipartite Number 111

	111	$\begin{matrix} 110 \\ 001 \end{matrix}$	$\begin{matrix} 101 \\ 010 \end{matrix}$	$\begin{matrix} 011 \\ 100 \end{matrix}$	$\begin{matrix} 100 \\ 010 \\ 001 \end{matrix}$
111	P_1				
$\begin{matrix} 110 \\ 001 \end{matrix}$	P_2	P_{11}			
$\begin{matrix} 101 \\ 010 \end{matrix}$	P_2		P_{11}		
$\begin{matrix} 011 \\ 100 \end{matrix}$	P_2			P_{11}	
$\begin{matrix} 100 \\ 010 \\ 001 \end{matrix}$	P_3	P_{21}	P_{21}	P_{21}	P_{111}

TABLE III—*Continued*
The Multipartite Number 22

	22	21 01	12 10	20 02	11 11	20 01 01	02 10 10	11 10 01	10 10 01 01
22	P_1								
21 01	P_2	P_{11}							
12 10	P_2		P_{11}						
20 02	P_2			P_{11}					
11 11	P_2				P_{11}				
20 01 01	P_3	$2P_{21}$		P_{21}		P_{111}			
02 10 10	P_3		$2P_{21}$	P_{21}			P_{111}		
11 10 01	P_3	P_{21}	P_{21}		P_{21}			P_{111}	
10 10 01 01	P_4	$2P_{31}$	$2P_{31}$	P_{22}	$2P_{22}$	P_{211}	P_{211}	$4P_{211}$	P_{1111}

Tables can be made for the partitions of the various multipartite numbers. In Table III are presented values for the numbers 11, 111, 22.

When the units are indistinguishable 11 condenses to the $w = 2$ part of Table I.

When the units are indistinguishable 111 condenses to the $w = 3$ part of Table I.

When the units are alike 22 condenses to the $w = 4$ part of Table I.

49. Multivariate Distributions. The chief results of Chapters II, III, IV also hold for multivariate distributions. Some additional definitions are neces-

sary. We suppose that the N variates $x_1, x_2, \dots, x_N$ are replaced by the Nr variates of the array

$$\begin{array}{ccccccc} {}_1x_1, {}_1x_2, & \cdots, & {}_1x_N \\ {}_2x_1, {}_2x_2, & \cdots, & {}_2x_N \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ {}_rx_1, {}_rx_2, & \cdots, & {}_rx_N \end{array} \quad \{78\}$$

where the presubscript represents the variable. The power sums become

$$\begin{aligned} (a_1) &= {}_1x_1^{a_1} + {}_1x_2^{a_1} + \cdots + {}_1x_N^{a_1} = \sum {}_1x_i^{a_1} \\ (a_2) &= {}_2x_1^{a_2} + {}_2x_2^{a_2} + \cdots + {}_2x_N^{a_2} = \sum {}_2x_i^{a_2} \end{aligned}$$

It is not necessary to utilize the presubscript since it is precisely the subscript of the a . That is the power sum (a_k) is defined by $\sum x_i^{a_k}$. Similarly $(a_1 a_2) = \sum_{i \neq j} {}_1x_i^{a_1} {}_2x_j^{a_2}$ can be written as $(a_1 a_2) = \sum_{i \neq j} x_i^{a_1} x_j^{a_2}$ without introducing ambiguity.

In general {6} as well as {4}, now holds for the multivariate case. It follows at once that the results of Chapters II, III, IV can be written for the multivariate case by means of the formulas of Chapter I as indicated by the previous section. Thus the formula for $P(1_1 1_1 1_2 1_2)$ may be written as [Table III]

$$\begin{aligned} P[\overline{10} \cdot \overline{10} \cdot \overline{01} \cdot \overline{01}] &= P_4 \overline{22} + 2P_{31} \overline{21} \cdot \overline{01} + 2P_{31} \overline{12} \cdot \overline{10} + P_{22} \overline{20} \overline{02} + 2P_{22} \overline{11} \overline{11} \\ &+ P_{211} \overline{20} \overline{01} \overline{01} + P_{211} \overline{02} \overline{10} \overline{10} + 4P_{211} \overline{11} \overline{10} \overline{01} + P_{1111} \overline{10} \overline{10} \overline{01} \overline{01} \end{aligned}$$

and can be interpreted as:

$$\begin{aligned} (10)^2(01)^2 &= (\overline{22}) + 2(\overline{21} \cdot \overline{01}) + 2(\overline{12} \cdot \overline{10}) + (\overline{20} \cdot \overline{02}) + 2(\overline{11} \cdot \overline{11}) + (\overline{20} \cdot \overline{01} \cdot \overline{01}) \\ &+ (\overline{02} \cdot \overline{10} \cdot \overline{10}) + (\overline{11} \cdot \overline{10} \cdot \overline{01}) + (\overline{10} \cdot \overline{10} \cdot \overline{01} \cdot \overline{01}) \end{aligned}$$

by {12} of Chapter II. It may also be interpreted as

$$\begin{aligned} (\overline{10} \cdot \overline{10} \cdot \overline{01} \cdot \overline{01}) &= -6(22) + 4(21)(01) + 4(12)(10) + (20)(02) \\ &+ 2(11)(11) - (20)(01)(01) - (02)(10)(10) \\ &- 4(11)(10)(01) + (10)(10)(01)(01) \end{aligned}$$

by {44} of Chapter II. It can also be interpreted as a double expansion by means of section 44 where the values of the P 's are given by the usual

$$\begin{aligned} P_r &= \sum_i (-1)^{\rho-1} (\rho-1)! \binom{1^r}{p_1^{\tau_1} \cdots p_s^{\tau_s}} k_{p_1^{\tau_1} \cdots p_s^{\tau_s}} \\ P_{r_1 \cdots r_g} &= \overline{P_{r_1} P_{r_2} \cdots P_{r_g}} \end{aligned}$$

50. Summary. It is apparent that {4} not only expresses (a) the number of ways in which the parts of one partition may be collected to form the parts

of another partition, (b) the formula for expanding products of power sums in terms of power product sums, (c) the formula expanding power product sums in terms of power sums, and (d) the formula for double expansions, but also that it can be used to make similar expansions in the case of multivariate distributions.

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ON COMBINED EXPANSIONS OF PRODUCTS OF SYMMETRIC POWER
SUMS AND OF SUMS OF SYMMETRIC POWER PRODUCTS
WITH APPLICATIONS TO SAMPLING (Continued)

By PAUL S. DWYER

PART II. THE FUNDAMENTALS OF SAMPLING

Introduction

We consider a population of N variates in which every individual possesses a common attribute. Let the variate x_i be the measure of such an attribute for individual i . From the N variates it is possible to form $\binom{N}{n}$ different samples where each sample consists of n variates, $n \leq N$.

Each sample has its mean, variance, etc. so that there are $\binom{N}{n}$ means, $\binom{N}{n}$ variances, etc. The fundamental sampling problem, as interpreted here, is to find the relation between the moments of the $\binom{N}{n}$ means, and the moments of the $\binom{N}{n}$ variances in terms of the moments of the universe. Numerous attempts have been made to solve this problem, but each has been restricted in some way. It is the aim of Part II to indicate an approach which is broad enough to include many of the fundamental variations.

The first chapter is devoted to a listing of criteria which should be satisfied by a theoretical development which is to be considered sufficiently general. These criteria might be applied to other statistics but the theory developed here is limited to those statistics which are moments (or functions of moments) of moments. The first chapter continues with an account of the more significant papers which have contributed to a general solution of the problem. No attempt is made to indicate a complete history, but rather there is presented a brief summary of a number of the most significant contributions.

The second chapter is devoted to definitions and notation. An attempt has been made to use conventional notation whenever it is suitable.

The third chapter deals with some of the fundamental principles which are used in the general approach. It presents a crucial part of the argument as it shows how various types of sampling problems can be reduced to Carver functions.

The last three chapters deal with specific applications to some of the simpler problems. Chapter IV discusses the case of moments of the mean of the sample.

Chapter V considers the mean of the variance and the variance of the variance, while Chapter VI gives a large number of formulas, implicitly, in tabular form.

Chapter I. A Brief History of Previous Contributions

In order to assist the reader in getting a perspective with reference to previous mathematical work on the relations between the moments of the moments of the sample and the moments of the complete set of measures (universe), a list of criteria¹ is suggested below which might be applied to each contribution. These criteria group themselves naturally into two classes. The first eight questions can be answered categorically, while the remainder are less definite in nature and are not so subject to categorical answers.

1. **The Criteria.** 1. Does the method apply to one type of frequency distribution only or is it broad enough in scope to include any distribution law?

2. Is there any restriction as to the size of the sample?

3. Is there any restriction as to the size of the universe?

4. Is there any restriction as to the nature of the correlation between observations? More specifically, is the method applicable only to some particular law of formation of the sample such as "drawing with replacements," "drawing without replacements," etc., or is it broad enough in scope to allow application to other orderly replacement laws?

5. Is the application limited to one characteristic (variable) or can a large number of characteristics be treated simultaneously?

6. Is it necessary that the universe maintain the same frequency distribution during the formation of the sample or may it assume a different frequency distribution before each drawing?

7. Does the method produce exact, rather than approximate, formulas?

8. Does the method permit approximations to a required degree of accuracy?

9. Does the method enable the author to write general laws in a compact form? More specifically, can he express, in a form which is not too symbolic, any moment of a given sample moment? If not, what order of moments can be expressed?

10. Is the notation such that the general case can be turned into the more important special cases with relative ease?

11. Does the development lead logically to the introduction of new moment functions (such as the semi-invariant of Thiele [B'; 209] or the k functions of R. A. Fisher [23; 203]) which are useful in condensing the results?

12. Is a combinatorial analysis provided so that any given formula, or any part of it, can be checked for accuracy without too much effort?

2. **Review of previous results.** The articles below have been examined with the criteria in mind. No attempt is made to write specific answers to all the

¹Many of these criteria have been suggested, in less explicit form by Tchouproff (15; 461-471). The "Introduction" of his Metron paper is recommended for use as a supplement to the present chapter.

criteria in each case, but rather to indicate the important features of each contribution.

The papers discussed by no means cover all the work on moments of moments, although a rather complete bibliographical background is available to the reader who desires to examine the bibliographies attached to the articles mentioned. Undoubtedly the importance of the articles written in English has been over-emphasized. Since the important contributions of non-English writers (such as Thiele and Tchouproff) have eventually appeared in English, it does no serious harm to refer to the English versions even though the results may have been partially antedated by the author in some other language.

A large number of the earlier results on moments of moments were limited to a special case of the problem, usually the case in which the universe is infinite and normal. The present summary deals with those authors who, during the past four decades, have made real contributions to the problem of generalization. A detailed account of the history of moments of moments would include many valuable contributions which are not included here.

It seems expedient to start with Pearson's article "On the Probable Error of Frequency Constants" [2] which appeared at the opening of the century. Although by no means the first article in the field, it presented a rather complete set of formulas for the case of moments of moments. One advantage of these formulas is that they are relatively brief and yet this brevity results from the fact that they are approximate. The original paper dealt with the univariate case, but it was followed by a later one [6] which discussed the case of more than one variable.

These formulas have played an important rôle in that they have assisted in making it clear that the moments of moments of samples must be estimated if one is to be permitted to draw conclusions from his sampling moments and that it is possible to work out formulas which serve as the basis of those estimates.

Of great importance also was the contribution of T. N. Thiele to the sampling problem. Adapting certain ideas of Laplace, he used semi-invariants in which to express his results which he published in English in 1903 in "The Theory of Observations" [B'; 209]. He took the case of the infinite parent and any law of distribution and then worked out moments through the fourth of the variance.

An earlier contribution of the introductory period was that of Karl Pearson in 1899 [1]. This paper is significant in that it provides formulas for the four moments of the mean when sampling is from a *finite* universe. The universe is not general, but obeys a simple frequency law.

Another article of this period was that of Robert Henderson (1904) in which the first four moments of the mean were given for an infinite universe with any frequency law. This article, which was first published in *Transactions of the Actuarial Society of America* [3], was considered so important that it was re-published in 1907 in the *British Journal of the Institute of Actuaries*. Henderson gave, in addition to the first four moments of the mean, first moments of m_2 , m_3 , m_4 although the last of these formulas is erroneous.

Another important contribution of this period was that of "Student" in 1908 [5]. He was interested in the properties of the normal distribution, but did not assume normality in his general derivation. He took an infinite population and wrote the formula for the variance of the variance. In this result he inserted the condition for normality. His further argument in the normal case implied the development of corresponding formulas for the higher moments of the variance, but he did not publish them as they were incidental to his main attack. The semi-invariant equivalent of these results had been previously given by Thiele [B'; 209-210].

The real contribution of "Student" to the general problem of moments of moments was his method, for it is his method which has been utilized by later writers. "Student's" method has the advantage that the development involves algebraic processes only. Contributions of Neyman, Church, Pepper, Carver, and the present writer are based upon it.

An important development during the next decade, 1908-1918, was the establishment of the first four moments of the mean when the samples were drawn from a finite parent without replacement. It appears that a number of men worked this problem independently. For example, one might examine the results of Pearson [4], Isserlis [7, 8], Mortara [C], Tchouproff [11], and Edgeworth [9]. Probably the best English presentations of that era were those of Isserlis [8] and Edgeworth [9] which appear in the same volume of the *Journal of the Royal Statistical Society*.

A most prolific writer on sampling during the next decade was the Russian, Tchouproff, who had been publishing in Russian and Scandinavian journals [10], [11]. His most valuable contributions were published in 1918-1923 in *Biometrika* (in English) and in *Metron* (in English).

The first series of articles was published in three different numbers of *Biometrika* in the years 1918-19 [12]. Tchouproff assumed an infinite universe and used the method of mathematical expectation. At first glance the most characteristic aspect of his work appears to be the complicated notation which he used. This notation was adopted because he undertook a much more general problem than had previously been attempted and hence needed to make new distinctions. Although he limited himself to the infinite case and one variable, he worked out the theory with the freedom that the frequency distribution of the universe might change between drawings. In the special case in which the populations are the same, he worked out the moments of the variance as far as the fourth. The chief criticism of his work concerns the complicated notation which seems to have been difficult to follow critically. A mistake in one of his formulas was not discovered for some years and then not by examination of his reasoning, but through the application of his results to an actual problem [17].

It is perhaps appropriate to insert here that in 1934 Feldman [30] rewrote the material of the second *Biometrika* article by simplifying the notation and extending the argument to the case of two (and more) variables.

Tchouproff continued to generalize his work and in the 1923 volume of *Metron*

[15] there appeared a series of articles in which there were no restrictions as to the size of the sample, no restrictions as to the type of sampling distribution (in fact the sampling distribution might vary between successive drawings), and no restrictions as to the law of replacement, or more generally as he expressed it, "no restriction as to the nature of the correlation between observations." Criterion number 5 is the only one of the first eight criteria which is not satisfied in as much as the approach is limited to that of a single variable. Also the notation was extremely complicated and, although Tchouproff gave general formulas for moments of moments, these formulas are so symbolic in form that he did not find it expedient to write out specific formulas beyond the variance of the variance for such an important special case as sampling from a finite parent without replacements.

During the same period J. Splawa-Neyman [14] had been examining the problem of sampling from a finite parent without replacements. He published his results in a Polish journal in 1923 [14] and his corrected results two years later in *Biometrika* [18]. He gave the well known formulas for the first four moments of the mean and a formula for the variance of the variance. He also gave some simple correlation formulas such as the correlation between the mean and the variance.

At this time the basic problem of moments of moments, at least as it was interpreted by Pearson and his followers, was the establishment of the first four moments of the given moment of the sample so that a Pearson curve could be fitted. A. E. R. Church, a worker in Pearson's laboratory, was assigned the task of seeing how the moments of the variance work out in actual practice. In doing this he became convinced that the formula for the fourth power of the variance, which had appeared in Tchouproff's *Biometrika* article, was incorrect. He tried to follow the argument of Tchouproff, but apparently was baffled by the complex notation and finally, at the suggestion of Pearson, decided to carry through the formula using the method of "Student." In doing this he discovered a mistake in the Tchouproff formula for the fourth power of the variance. At the same time he published [17] the formulas for the third and fourth power of the variance in the more conventional notation of that time.

It might be noted that it is particularly fitting that Church should discover this error since Tchouproff, as Pearson himself stated in an editorial [13], had pointed out a number of errors made by the Pearsonian school.

In the next volume of *Biometrika* there appears an article by Church [19] in which, among other things, formulas are derived for the third and fourth moments of the variance in the case of a finite population, sampling without replacement. Church claimed no particular credit for these formulas. His point is rather that they are almost valueless from a practical standpoint chiefly because of their length. The formula for the fourth power of the variance occupies three and one-half of the large pages of *Biometrika* and is given with the apparent aim of indicating, as Pearson said [21; 209], "the practical futility of the theoretical formulas."

Church gave full credit to Neyman for the formula for the variance of the variance and made no mention of Tchouproff's *Metron* work and of the more general presentation there given. This was particularly unfortunate because it exposed him to the charge that he ignored non-English authors. This charge was immediately made by Greenwood and Isserlis [20] who broadened it to include Neyman and, by implication, Pearson himself. They advocated the case of Tchouproff who, now dead, was unable to defend himself. They gave a survey (valuable to the cursive reader) of the pertinent contributions of the Tchouproff articles and suggested that the ignoring of Tchouproff was particularly disconcerting since it appears that Tchouproff had gone more than half way in his cooperation with English writers.

Pearson replied in an interesting article [21] which made it clear that Neyman established his results independently of Tchouproff and that the language of Neyman is much simpler than the complicated notation of Tchouproff. Pearson emphasized that Tchouproff made no attempt to give specific formulas for the third and fourth moments of the variance in the case of sampling with replacements. Pearson did not answer, at least explicitly, the claim that the Tchouproff formulas are applicable to a more general case in which there is no restriction as to the nature of the correlation between observations.

The year 1928 was marked by two important contributions. We first mention that of C. C. Craig who published his thesis in *Metron* [22]. Extending the previous results of Thiele, he was able to write the semi-invariant equivalent of the basic formulas in much less space than their previous moment formulation had demanded. He was able to write products of sample moments as well as moments of the moments themselves. His results are limited to an infinite population and one variable. The bibliography attached to his paper is commonly mentioned in later literature for its completeness. For infinite sampling it might properly be used as a supplement to the bibliography of this Part.

A most important contribution was made by R. A. Fisher [23] who was able to simplify the infinite sampling formulas greatly. He did this by introducing the sample function whose expected value is a cumulant (semi-invariant). In addition to the simplification, his ingenious attack resulted in the following contributions: (1) the recognition of the one to one correspondence between all possible independent sampling formulas and the partition of numbers, (2), that the extension of the multivariate form is accomplished by use of the partitions of multipartite numbers, (3) the tabulation of numerous new formulas, (4) the use of a general partition method by which any term in the formulas can be determined separately.

The further development of the combinatorial analysis was indicated by a paper by Fisher and Wishart which appeared in 1931 [27]. It was shown how the more involved patterns could be broken up into simpler ones.

The study of the infinite case was continued by Georgescu [28] who extended the Craig results. A feature of his work was the utilization of functions which yielded expansions of formulas in terms of successive degrees of approximation.

He applied Fisher's idea of a combinatory analysis to the conventional sample moment function.

Another paper of this series was that of Wishart [29] who gave a descriptive account of the contributions of Craig, Fisher, and Georgescu and an indication of the means of expressing the results of one writer into the language of another.

The work of Joseph Pepper which appeared in *Biometrika* in 1929 [24] should be noted. Pepper took the case of the finite parent, sampling without replacement, and two variables, and then gave an extensive list of results. He did not have a very condensed notation and was forced to assume an infinite universe for the higher moments which he studied. The important point, for historical purposes, is that Pepper combined bivariate and finite sampling. It is to be recalled that Tchouproff himself in his generalized theory gave no results for the multivariate case.

A significant advance in finite sampling was indicated by the appearance of Carver's editorial on "Fundamentals of the Theory of Sampling," which appeared in the first volume of the *ANNALS OF MATHEMATICAL STATISTICS* [25]. Carver took the case of a finite universe, one variable, and sampling without replacements. He presented a notation which enabled him to write the various moments of the mean through the eighth in simple form. He showed by a number of illustrations that his formula would give known results for cases both infinite and finite, when the proper restrictions were added. O'Toole [26] later generalized his results for any moment of the mean.

3. Generalized Carver Functions and Sampling. The use of generalized Carver functions together with the results of Part I makes possible the presentation of the general sampling theory in a compact, and yet not too symbolic, form. It is possible to write the sampling theory so that criteria 1-8 are satisfied although no attempt is made in the present paper to answer criterion 6. With reference to criteria 9-11, any affirmative answer must necessarily be tempered with qualifications as the results are far removed from that ideal solution which would permit one to determine the actual distribution of any sample moment. However the use of generalized Carver functions does permit a general concise statement of results as well as the determination of special cases. The method is also especially adapted to the introduction of new moment functions and to the use of partition analysis, although these topics are not emphasized in the present paper. In general it may be said that the use of Carver functions assists greatly in finding the theoretical sample statistics in the case of finite sampling since the Carver functions are condensed expressions of the size of the sample and the size of the parent, since they may be easily checked from symmetrical considerations, and since they are independent of the moments. They are also applicable to different replacement laws.

4. The Use of High Moments. Precise agreement between theoretical and practical sampling does not usually accompany the use of high moments, and

the practical statistician is apt to agree with Pearson who wrote, "I have a very firm conviction that the mathematician who uses high moments may make interesting contributions to mathematics, but he removes his work from any contact with actual statistics" [16; 117]. However since the extent of agreement between theoretical and actual results is in a sense a measure of the extent to which theoretical assumptions are actually duplicated in the experiment, it does seem sensible to discover what relations exist in the ideal theoretical case. Thiele implicitly supported the theoretical use of high moments (even in studying actual problems) when he wrote [B'; 13]:

"Therefore the general rule of the formation of good laws of presumptive errors must be:

1. In determining λ_1 , and λ_2 rely almost entirely upon the actual values.
2. As to the half-invariants with high indices, say λ_6 upwards, rely as exclusively upon theoretical considerations.
- 3"

A more explicit advocate is R. A. Fisher who wrote [23; 200], "In the present state of our knowledge any information, however incomplete, as to sampling distributions is likely to be of frequent use, irrespective of the fact that moment functions only provide statistical estimates of high efficiency for a special type of distribution."

Chapter II. Notation and Definition

The present chapter gives the fundamental definitions and appropriate notation. An attempt has been made to combine the most desirable features of the different notations of earlier writers.

5. Ordered Sample. An ordered sample is a sample in which distinction is made as to the order in which the variate enters the growing sample. Thus the sample found by drawing x_2 and then x_1 is the same sample as that obtained by drawing x_1 and then x_2 , but it is a different ordered sample.

In some types of sampling it is possible that a given variate may appear more than once in the same sample. In general the number of ordered samples varies with the number of repeated variates. Thus the sample $x_1 + x_1$ results from but one ordered sample, while $x_1 + x_2$ results from either of two ordered samples.

6. Power Sums. Power sums have the same meaning as in section 11 of Part I. An adjustment of notation is necessary as we need to distinguish power sums of the sample from power sums of the universe. The a -th power sum of the universe is denoted by (A) while the sample power sum is denoted by (a) . Similarly, bold-faced numerals are used to indicate power sums of the universe, while light-faced numerals are used to indicate power sums of the sample. The symbol $(\bar{A})$ is used to indicate that the variates are deviations from the mean of the universe.

7. Power Product Sums. Power product sums, called power products for brevity, also have the same meaning as in section 11 of Part I. Large letters are used to represent the power products of the universe while small letters are used to indicate the power products of the sample. Thus $(Q_1 Q_2 \cdots Q_s)$ represents a power product of the universe while $(q_1 q_2 \cdots q_s)$ represents the corresponding power product of the sample. Power products are not used extensively except in the development of the theory of the next chapter where they play an important rôle.

8. Expected Values. If a given statistical function, z is formed for every possible sample, then the arithmetic mean of the z 's is the expected value of z . Thus $E(z) = \frac{\sum (z)}{S}$ where the Σ holds for all possible samples and S is the number of such samples.

9. Moments. Moments demand precise notation since distinction must be made between moments of the universe, moments of the sample, moments of the moments of the sample, and moments about the mean for these cases. In addition we wish to indicate whether or not the universe is measured about its mean.

a. *Moments of the universe.* The conventional μ 's are used to indicate the moments of the universe. In this notation $\bar{\mu}$ is used to indicate the moment about the mean of the universe. Thus

$$\mu_t = \frac{(T)}{N} = \frac{\sum x^t}{N} \quad \text{and} \quad \bar{\mu}_t = \frac{(\bar{T})}{N} = \frac{\sum \bar{x}^t}{N}.$$

The usual formula relating μ_t and $\bar{\mu}_t$ [22; 20] may be written

$$\bar{\mu}_t = \sum (-1)^s \binom{1^t}{(t-s) \cdot 1^s} \mu_{t-s} \mu_1^s \quad \{1\}$$

so that

$$\begin{aligned} \bar{\mu}_2 &= \frac{(2)}{N} - \frac{(1)(1)}{N^2}, \\ \bar{\mu}_3 &= \frac{(3)}{N} - \frac{3(2)(1)}{N^2} + \frac{2(1)^3}{N^3}, \\ &\text{etc.} \end{aligned}$$

It is to be noted that, when $(1) = 0$, $\bar{\mu}_t = \mu_t$.

b. *Moments of the sample.* We denote the moments of the sample by the letter m [23; 203].

In much statistical work deviations from the mean of the universe are used in place of the variates themselves. When the universe moments about the mean appear, we indicate them with a bar. However in denoting the moments of the

samples, the moments of the mean do not appear and some other device is needed to indicate whether or not the variates are measured about the mean of the universe. The simple notations m_t and $\bar{m}_t$ are used to indicate that the variates used are deviations from the mean of the universe. A superprefix is used to indicate the case in which the variates are not measured about the mean, 1m_t , $^1\bar{m}_t$. The values of $\bar{m}_t$ (and $^1\bar{m}_t$) are obtained from the values of m_t (and 1m_t) by means of the formula

$$\bar{m}_t = \sum (-1)^s \binom{t}{t-s} \cdot 1^s m_{t-s} m_1^s. \quad \{2\}$$

c. *Moments of the moments of a sample.* Since there are many possible samples and since a given moment can be computed for each sample, it is possible to express the expected value of this moment and the expected value of any power of it. The μ 's are used for this purpose. Thus

$$\begin{aligned} \mu_r(m_t) &= E(m_t)^r \\ \mu_r(^1m_t) &= E(^1m_t)^r \\ \mu_r(\bar{m}_t) &= E(\bar{m}_t)^r \\ \mu_r(^1\bar{m}_t) &= E(^1\bar{m}_t)^r. \end{aligned} \quad \{3\}.$$

If the first one of equations {3} represents the whole group, then the values $\bar{\mu}_r(m_t)$, $\bar{\mu}_r(^1m_t)$, $\bar{\mu}_r(\bar{m}_t)$, and $\bar{\mu}_r(^1\bar{m}_t)$ are indicated by

$$\bar{\mu}_r(m_t) = \sum (-1)^s \binom{r}{r-s} \cdot 1^s \mu_{r-s}(m_t) \mu_1^s(m_t). \quad \{4\}.$$

d. *Moments of the product of the moments of a sample.* The term $\frac{\sum xy}{N}$ can be indicated by $E(xy) = \mu_{11}(x, y)$. Similarly the expected value of the product of m_a and m_b may be indicated by $E(m_a m_b) = \mu_{11}(m_a, m_b)$. In general

$$\mu_{r_1 r_2 \dots r_s}(m_{a_1}, m_{a_2}, \dots, m_{a_s}) = E(m_{a_1}^{r_1} m_{a_2}^{r_2} \dots m_{a_s}^{r_s}) \quad \{5\}$$

In the case of the product of sample moment functions, when the universe is not measured about its mean, it is preferable to use a single superprefix, associated with the μ instead of a number of them associated with each m function. Thus

$$\mu_{111}(^1m_a, ^1m_b, ^1m_c) = ^1\mu_{111}(m_a, m_b, m_c).$$

The usual laws for changing from moments to moments about the mean in the case of the multivariate distributions are available. Thus

$$\bar{\mu}_{11}(m_a, m_b) = \mu_{11}(m_a, m_b) - \mu_{10}(m_a, m_b)\mu_{01}(m_a, m_b). \quad \{6\}$$

$$\begin{aligned} \bar{\mu}_{111}(m_a, m_b, m_c) &= \mu_{111}(m_a, m_b, m_c) - \mu_{110}(m_a, m_b, m_c)\mu_{001}(m_a, m_b, m_c) \\ &\quad - \mu_{101}(m_a, m_b, m_c)\mu_{010}(m_a, m_b, m_c) \\ &\quad - \mu_{011}(m_a, m_b, m_c)\mu_{100}(m_a, m_b, m_c) \\ &\quad + 2\mu_{100}(m_a, m_b, m_c)\mu_{010}(m_a, m_b, m_c)\mu_{001}(m_a, m_b, m_c) \end{aligned} \quad \{7\}$$

etc.

10. Different Sampling Laws. For theoretical purposes, any law may be used in the formation of samples as long as it results in functions of all possible samples which are symmetric functions of the variates. Any uniform law of replacement satisfies this condition and hence might be used in forming samples. Most statisticians who have worked on the sampling problem have been content to assume one or the other of two replacement laws. Each of these is "natural," since it has wide application in the study of actual sampling.

The two types of sampling which have received general treatment are *sampling from an infinite universe with any law of replacement* and *sampling from a finite universe with a law of no replacements*. The results of the first type are also applicable to the case of *sampling from a finite universe when replacements are made after each drawing*. These two types of sampling have been characterized by the terms "sampling from an infinite universe," or "sampling from an unlimited supply" [25; 114] and "sampling from a finite universe" [17], or "sampling from a limited supply" [25; 101].

The theory of moments of moments for the first type of sampling has been developed to a high degree by such authors as Craig [22], Fisher [23], and Georgescu [28]. This extensive development has been due in part to the fact that the assumption of an infinite universe permits application of methods which are not applicable to the study of finite variation. The probability of getting a variate remains the same no matter what the law of replacement. The assumption of an infinite universe at first appears to make the results inapplicable to all actual problems where the universe is finite. However, if the universe is large, the assumption of infinite size does not greatly alter the results, although the extent of the change can not be determined without comparison with the results of finite sampling. A justification for the use of infinite sampling in actual finite sampling problems is based on the fact that the formulas resulting from sampling from a finite parent with replacements are the same as the infinite formulas. Hence the infinite results may be used to characterize finite sampling if *sampling is done with replacement after each drawing*. This clever scheme is somewhat invalidated, in actual sampling, because of the practicability of replacing and remixing after each drawing. Until someone demonstrates a technique which is practical and effective in securing randomness, it must be said that the value of infinite sampling theory as applied to finite

sampling depends upon the theoretically unsatisfactory assumption that a finite universe is infinite.

The theory of sampling from a finite universe without replacements has been developed by such authors as Isserlis [8], Tchouproff [15], Neyman [18], Church [19], Pepper [24], and Carver [25], although available results are not as extensive as those mentioned above because of the difficulty of algebraic manipulation and because of the length of the formulas. The fact is that the probability of getting a given variate varies with the different drawings. However, a "return to the bag" is not demanded.

The terms "infinite sampling" and "finite sampling" are adequate to describe the two kinds of sampling discussed above, but they are inadequate in the case of finite sampling if additional replacement laws are introduced. Hence, it seems preferable to characterize the type of sampling by the replacement law if the population is finite.

When the Carver functions represent known functions of n and N , it is possible to use them in writing moment formulas for any orderly replacement law. For example, it is shown in later sections how Carver functions can be applied to

1. Finite sampling without replacement,
2. Finite sampling with replacement after each drawing.
3. Finite sampling without replacement up to the n -th drawing before which the $n - 1$ withdrawn variates are replaced and mixed.

The Carver function can be used symbolically even in cases in which its explicit statement in terms of n and N has not been found. In some statistical formulas the Carver functions cancel, so that the results are independent of the sampling law.

11. Variable Distribution Laws. It is possible to generalize the theory to include the case in which the variable takes on a different frequency distribution after each drawing, i.e., the general Tchouproff formulas can be written in terms of Carver functions. This theory can also be generalized to include many variables. In this dissertation, however, it is assumed that the universe remains the same, aside from the unreplaced variates forming the sample, throughout the sampling process.

Chapter III. The Application of the Double Expansion Theorem

It is the purpose of this chapter to establish the basic theorems on which the more specific work of the later chapters is based and to show how the double expansion theorem is to be applied to the sampling problem.

12. Formulas Concerning Ordered Samples. a. *Sampling with replacements.* If the samples of n are taken from a universe of N variates and if the variates are replaced after each drawing, then the number of possible ordered samples is N^n since for each of the n drawings there is a choice of N .

b. *Sampling without replacement.* If the variate is not replaced after each drawing, the number of ordered samples is

$$N(N-1) \cdots (N-n+1) = N^{(n)}.$$

c. *Replacement before the last drawing only.* In case sampling is with replacement before the last drawing only, the number of ordered samples is

$$N(N-1) \cdots (N-n+2)N = N^{(n-1)}N.$$

13. Theorem I. *All moments of moment functions of samples can be expressed in terms of expected values of products of power sums of samples.*

By moment functions we mean rational integral isobaric moment functions [31; 22].

The theorem follows at once from the definitions of section 9. From $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$, $\{7\}$ it is clear that all moments of moment functions of samples are expressible in terms of the expected values of sample moment functions. But since the sample moment functions are themselves defined in terms of power sums of the samples, the theorem follows. For example

$$\bar{\mu}_2(\bar{m}_2) = \mu_2(\bar{m}_2) - \mu_1^2(\bar{m}_2) = E\left[\frac{(2)}{n} - \frac{(1)(1)}{n^2}\right] - \left[E\left\{\frac{(2)}{n} - \frac{(1)(1)}{n^2}\right\}\right]^2 \quad \{8\}$$

and

$$\begin{aligned} \bar{\mu}_{11}(\bar{m}_2, m_1) &= \mu_{11}(\bar{m}_2, m_1) - \mu_{10}(\bar{m}_2, m_1)\mu_{01}(\bar{m}_2, m_1) \\ &= E\left[\frac{(2)(1)}{n^2} - \frac{(1)^3}{n^3}\right] - \left[E\left\{\frac{(2)}{n} - \frac{(1)(1)}{n}\right\}\right]\left[E\left\{\frac{(1)}{n}\right\}\right] \quad \{9\} \end{aligned}$$

14. Theorem II. *All moments of moment functions of samples can be expressed in terms of expected values of power products of samples.*

This follows at once from the application of the multiplication theorem of Part I to the theorem of section 13. Each product of power sums is expanded by the multiplication theorem into sums of power products. Thus

$$\begin{aligned} \mu_2(\bar{m}_2) &= E\left[\frac{(2)(2)}{n^2} - \frac{2(2)(1)(1)}{n^3} + \frac{(1)^4}{n^4}\right] \\ &= \left(\frac{1}{n^2} - \frac{2}{n^3} + \frac{1}{n^4}\right)E(4) + \left(-\frac{4}{n^3} + \frac{4}{n^4}\right)E(31) + \left(\frac{1}{n^2} - \frac{2}{n^3} + \frac{3}{n^4}\right)E(22) \\ &\quad + \left(-\frac{2}{n^3} + \frac{6}{n^4}\right)E(211) + \frac{1}{n^4}E(1111). \quad \{10\} \end{aligned}$$

15. Theorem III. *To every power product form $(q_1 q_2 \cdots q_s)$ there corresponds a power product form $(Q_1 Q_2 \cdots Q_s)$.*

The argument is simple since the terms of $(q_1 q_2 \cdots q_s)$ are themselves terms of $(Q_1 Q_2 \cdots Q_s)$. It follows at once that, if $(q_1 q_2 \cdots q_s)$ exists, then $(Q_1 Q_2 \cdots Q_s)$ exists.

As an illustration, consider the universe consisting of x_1, x_2, x_3, x_4, x_5 and the sample consisting of x_1, x_2, x_3, x_4 . Then the terms of $(q_1 q_2 q_3) = \sum_{i_1 \neq i_2 \neq i_3}^4 x_{i_1}^{q_1} x_{i_2}^{q_2} x_{i_3}^{q_3}$ are all contained in the terms of $(Q_1 Q_2 Q_3) = \sum_{i_1 \neq i_2 \neq i_3}^5 x_{i_1}^{q_1} x_{i_2}^{q_2} x_{i_3}^{q_3}$.

16. **Theorem IV.** *If definite k 's can be determined so that*

$$E(q_1 q_2 \cdots q_s) = k_{p_1 p_2 \cdots p_s} (Q_1 Q_2 \cdots Q_s), \quad \{11\}$$

then it is possible to use the double expansion theorem and express the moments of the moments of the sample in terms of the P functions of Part I and the power sums (or moments) of the universe.

The double expansion theorem was designed to replace $(q_1 q_2 \cdots q_s)$ by $k_{p_1 p_2 \cdots p_s} (Q_1 Q_2 \cdots Q_s)$. It can be used as well to replace $E(q_1 q_2 \cdots q_s)$ by $k_{p_1 p_2 \cdots p_s} (Q_1 Q_2 \cdots Q_s)$ if the values of $k_{p_1 p_2 \cdots p_s}$ can be determined. The results of such a substitution in terms of the power sums of the universe are then given by the double expansion theorem. For example

$$\mu_2({}^1 m_1) = \frac{E(1)^2}{n^2} = \frac{E(2)}{n^2} + \frac{E(11)}{n^2}$$

and if $E(2) = k_1(2)$ and $E(11) = k_{11}(11)$ then

$$\begin{aligned} \mu_2({}^1 m_1) &= (k_2 - k_{11}) \frac{(2)}{n^2} + \frac{k_{11}(1)^2}{n^2} \\ &= \frac{K_2}{n^2} (2) + \frac{K_{11}(1)^2}{n^2} \end{aligned}$$

where $K_2 = k_2 - k_{11}$ and $K_{11} = k_{11}$.

It then appears that the methods and tables of Chapter I of Part I can be used in finding expressions for moments of moments, in case $k_{p_1 p_2 \cdots p_s}$ is known. Thus

$$\begin{aligned} \mu_2({}^1 \bar{m}_2) &= E \left[\frac{(2)}{n} - \frac{(1)(1)}{n^2} \right]^2 = E \left[\frac{(2)(2)}{n^2} - \frac{2(2)(1)(1)}{n^3} + \frac{(1)(1)(1)(1)}{n^4} \right] \\ &= \frac{P_2(4) + P_{11}(2)(2)}{n^2} - 2 \left[\frac{P_3(4) + 2P_{21}(3)(1) + P_{21}(2)(2) + P_{111}(2)(1)(1)}{n^3} \right] \\ &\quad + \frac{P_4(4) + 4P_{31}(3)(1) + 3P_{22}(2)(2) + 6P_{211}(2)(1)(1) + P_{1111}(1)^4}{n^4} \end{aligned}$$

and when $(1) = 0$

$$\mu_2({}^1 \bar{m}_2) = \left(\frac{P_2}{n^2} - \frac{2P_3}{n^3} + \frac{P_4}{n^4} \right) (\bar{4}) + \left(\frac{P_{11}}{n^2} - \frac{2P_{21}}{n^3} + \frac{3P_{22}}{n^4} \right) (\bar{2})(\bar{2}). \quad \{13\}$$

where

$$\begin{aligned}
 P_4 &= k_4 - 4k_{31} - 3k_{22} + 12k_{211} - 6k_{1111} \\
 P_{31} &= k_{31} - 3k_{211} + 2k_{1111} \\
 P_{22} &= k_{22} - 2k_{211} + k_{1111} \\
 P_{211} &= k_{211} - k_{1111} \\
 P_{1111} &= k_{1111}
 \end{aligned}$$

as given by {54} of Part I.

The basic problem has thus been reduced to finding $k_{p_1 \dots p_s}$ such that

$$E(q_1 q_2 \dots q_s) = k_{p_1 \dots p_s} (Q_1 Q_2 \dots Q_s).$$

17. Theorem V. *The expected value of a sample power sum is always $\frac{n}{N}$ times the corresponding universe power sum no matter what the replacement law.*

The expected value of the sample power sum is always the same even though the k 's take on different values for different replacement laws. We note first that the number of ordered samples, S , depends upon the replacement law. Now a given sample power sum, (a) , has n terms, while the corresponding power sum of the universe, (A) , has N terms. All the a -th powers of the variates in the universe appear in the ordered samples and, if we add all possible ordered samples, these terms appear the same number of times. Hence

$$\sum (a) = k_1^{\downarrow}(A) \quad \text{and} \quad \frac{\sum (a)}{(A)} = k_1^{\downarrow}.$$

Now the number of the a -th powers of the variate in $\sum (a)$ is Sn so that each of the N variates appears $\frac{Sn}{N}$ times. It follows that $\sum (a) = \frac{Sn}{N}(A)$ and hence that $E(a) = \frac{n}{N}(A)$. Hence

$$E(a) = k_1(A) \quad \text{where} \quad k_1 = \frac{n}{N} \quad \{15\}$$

no matter what the law of replacement.

An illustration may serve to clarify the argument. Consider a universe composed of x_1, x_2, x_3 and write the six ordered samples. Then

$$\frac{\sum (a)}{(A)} = \frac{x_1^a + x_2^a + x_1^a + x_3^a + x_2^a + x_1^a + x_2^a + x_3^a + x_3^a + x_1^a + x_3^a + x_2^a}{x_1^a + x_2^a + x_3^a} = 4$$

and

$$\frac{E(a)}{(A)} = \frac{2}{3} = \frac{n}{N}.$$

18. **Value of $k_{p_1 \dots p_s}$ for sampling without replacement.** Consider a universe and all possible ordered samples. Form $(Q_1 Q_2 \dots Q_s)$ and $\sum (q_1 q_2 \dots q_s)$. Now $\sum (q_1 q_2 \dots q_s)$ is a symmetric function of the variates and consists of $N^{(n)} n^{(s)}$ products, and $(Q_1 Q_2 \dots Q_s)$ consists of $N^{(s)}$ products. Each of the $N^{(s)}$ products is repeated the same number of times in the $N^{(n)} n^{(s)}$ products of $\sum (q_1 q_2 \dots q_s)$. To find the number of times such repetition is made, it is only necessary to divide the total number of terms in $\sum (q_1 q_2 \dots q_s)$ by the number of terms in $(Q_1 Q_2 \dots Q_s)$ which gives $\frac{N^{(n)} n^{(s)}}{N^{(s)}}$. Hence

$$\sum (q_1 q_2 \dots q_s) = \frac{N^{(n)} n^{(s)}}{N^{(s)}} (Q_1 Q_2 \dots Q_s) \quad \{16\}$$

and, dividing by the number of ordered samples, $N^{(n)}$,

$$E(q_1 q_2 \dots q_s) = \frac{n^{(s)}}{N^{(s)}} (Q_1 Q_2 \dots Q_s) \quad \{17\}$$

so that

$$k_{p_1 \dots p_s} = \frac{n^{(s)}}{N^{(s)}} \quad \{18\}$$

as stated in section 46 of Part I.

Since $(q_1 q_2 \dots q_s) = s_1! s_2! \dots s_p! M(q_1 q_2 \dots q_s)$

and $(Q_1 Q_2 \dots Q_s) = s_1! s_2! \dots s_p! M(Q_1 Q_2 \dots Q_s)$

it follows that

$$EM(q_1 q_2 \dots q_s) = \frac{n^{(s)}}{N^{(s)}} M(Q_1 Q_2 \dots Q_s). \quad \{19\}$$

Most earlier writers on finite sampling have used the idea expressed in {19} as the foundation of their work. They have found it necessary to undertake enormous algebraic manipulation to expand in terms of monomial symmetric functions and then to expand back in terms of power sums after making the coefficient adjustment. Such long derivations are not only laborious, but they are also apt to result in algebraic errors and the results obtained have not emphasized the symmetry which is inherent in the nature of the problem and which is very useful in checking calculations. It was Carver who first discovered the type of symmetric relation involved and who used it in obtaining a compact statement of the first eight moments of the sample sum in the case of a single variable. He, too, found it necessary to carry out extensive algebraic manipulations as his reference to "lavish use of symmetric functions" [25; 104] reveals. His keen insight into the essential nature of this problem led him to the conclusion that such extensive algebraic manipulation should not be necessary and that it should be possible to apply P functions to sample moments of order higher than the first. His confidence that this could be done and his

encouragement in the task have contributed in a large degree to whatever merit this dissertation may have.

With $k_{p_1 \dots p_s} = \frac{n^{(s)}}{N^{(s)}}$, it is at once possible to write the P function expansions.

Following Carver, we let $\rho_1 = \frac{n}{N}$, $\rho_2 = \frac{n(n-1)}{N(N-1)}$, etc. and get, from sections 43 and 44 of Part I,

$$\begin{array}{ll} P_1 = \rho_1 & P_{11} = \rho_2 \\ P_2 = \rho_1 - \rho_2 & P_{21} = \rho_2 - \rho_3 \\ P_3 = \rho_1 - 3\rho_2 + 2\rho_3 & P_{31} = \rho_2 - 3\rho_3 + 2\rho_4 \\ P_4 = \rho_1 - 7\rho_2 + 12\rho_3 - 6\rho_4 & P_{22} = \rho_2 - 2\rho_3 + \rho_4 \\ \text{etc.} & \text{etc.} \end{array}$$

19. Expected Values of Products of Sample Power Sums, Sampling Without Replacement. The tables of Chapter I of Part I are now available for use. Thus

$$\mu_3({}^l m_1) = E({}^l m_1)^3 = \frac{1}{n^3} E(1)^3 = \frac{1}{n^3} [P_3(\mathbf{3}) + 3P_{21}(\mathbf{2})(\mathbf{1}) + P_{111}(\mathbf{1})^3]. \quad \{20\}$$

where

$$\begin{aligned} P_3 &= \frac{n}{N} - \frac{3n(n-1)}{N(N-1)} + \frac{2n(n-1)(n-2)}{N(N-1)(N-2)} \\ P_{21} &= \frac{n(n-1)}{N(N-1)} - \frac{n(n-1)(n-2)}{N(N-1)(N-2)} \\ P_{111} &= \frac{n(n-1)(n-2)}{N(N-1)(N-2)}. \end{aligned}$$

Formula {20} might be written as

$$\mu_3({}^l m_1) = \frac{1}{n^3} [P_3 N \mu_3 + 3P_{21} N^2 \mu_2 \mu_1 + P_{111} N^3 \mu_1^3] \quad \{21\}$$

We note further that as $N \rightarrow \infty$

$$NP_3 \rightarrow n, \quad P_{21}N^2 \rightarrow n(n-1), \quad P_{111}N^2 \rightarrow n(n-1)(n-2)$$

so that

$$\mu_3({}^l m_1) = \frac{1}{n^3} [n\mu_3 + 3n(n-1)\mu_2\mu_1 + n(n-1)(n-2)\mu_1^3] \quad \{22\}$$

More generally

$$P_{m_1 \dots m_r}(Q_1)(Q_2) \dots (Q_r) = P_{m_1 \dots m_r} N^r \mu_{q_1} \mu_{q_2} \dots \mu_{q_r}. \quad \{23\}$$

As N approaches infinity this becomes

$$P_{m_1 \dots m_r}(Q_1)(Q_2) \dots (Q_r) = n^{(r)} \mu_{q_1} \mu_{q_2} \dots \mu_{q_r}. \quad \{24\}$$

The laws of infinite sampling may be obtained by replacing power sums by moments and $P_{m_1 \dots m_r}$ by $n^{(r)}$. The tables given in a recent paper [31; 30-32] were obtained from the tables of P functions by this method.

20. Sampling With Replacements. We next consider the case of finite sampling with replacements after each drawing. This is such a simple case that the P 's can be determined without finding the k 's.

Consider a universe and the N^r possible ordered samples. Thus the nine ordered samples of 2 from a universe of 3 are indicated by the subscripts

11	21	31
12	22	32
13	23	33

The samples 11, 22, 33 are not repeated while the others are. The multiplication theorem can be used in grouping types of product terms as it was in Part I, but the terms themselves have different interpretation. Thus $(1)(1) = (2) + (11)$ can be written as $(1)(1) = (2) + [11]$ where the (2) indicates the sum of the n terms found by multiplying an x by itself, while the $[11]$ indicates the sum of the $n(n-1)$ products formed by multiplying one x by another. Since some of the x 's may be alike, it is possible to have squared terms in $[1 \cdot 1]$, but they are not treated as squared terms, but rather as products. For example, if $(1) = x_1 + x_1$

$$(1)(1) = x_1^2 + x_1^2 + x_1x_1 + x_1x_1$$

so that

$$(2) = x_1^2 + x_1^2 \text{ and } [11] = x_1x_1 + x_1x_1.$$

In determining the expected value of $(1)(1)$, we note that

$$\sum (1)(1) = \sum (2) + \sum [1 \cdot 1]$$

where $\sum$ holds for the N^n possible samples. Now $\sum (2) = k'_1(2)$ and $k'_1 = \frac{nN^n}{N}$

so that $E(2) = \frac{n}{N}(2)$ as indicated in Theorem V. Also $[11]$ is composed of $N^n n^{(2)}$ products of $\sum_{i,j=1}^N x_i x_j = \left(\sum_{i=1}^N x_i \right) \left(\sum_{j=1}^N x_j \right)$. It follows that

$$\sum [1 \cdot 1] = \frac{N^n n^{(2)}}{N^2} (1)(1) \text{ and that}$$

$$E[1 \cdot 1] = \frac{n^{(2)}}{N^2} (1)(1).$$

It appears that $\frac{n^{(2)}}{N^2}$ plays the rôle of P_{11} .

$$\begin{aligned}\mu_2(^1m_1) &= \frac{1}{n^2} E[(2) + [11]] \\ &= \frac{1}{n^2} [P_2(2) + P_{11}(1)(1)]\end{aligned}$$

where $P_2 = \frac{n}{N}$ and $P_{11} = \frac{n^{(2)}}{N^2}$.

The corresponding argument holds for the general case. Any product of power sums can be expanded in terms of $(q_1 q_2 \cdots q_s)$. If duplicate variates are introduced, use the notation $[q_1 q_2 \cdots q_s]$. Form $[q_1 q_2 \cdots q_s]$ for all the N^n ordered samples. Now $[q_1 q_2 \cdots q_s]$ has $n^{(s)}$ terms and $\sum [q_1 q_2 \cdots q_s] = k(Q_1)(Q_2) \cdots (Q_s)$ has $n^{(s)}N^n$ terms, while $(Q_1)(Q_2) \cdots (Q_s)$ has N^s terms.

It follows that $k = \frac{n^{(s)}N^n}{N^s}$, that

$$\sum [q_1 q_2 \cdots q_s] = \frac{n^{(s)}N^n}{N^s} (Q_1)(Q_2) \cdots (Q_s),$$

and that

$$E[q_1 q_2 \cdots q_s] = \frac{n^{(s)}}{N^s} (Q_1)(Q_2) \cdots (Q_s). \quad \{25\}$$

Hence

$$P_{m_1 \cdots m_s} = \frac{n^{(s)}}{N^s}. \quad \{26\}$$

In general

$$P_{m_1 \cdots m_s}(Q_1)(Q_2) \cdots (Q_s) = n^{(s)} \mu_{q_1} \mu_{q_2} \cdots \mu_{q_s}. \quad \{27\}$$

Comparison with {24} shows that the same basic laws appear no matter whether sampling is carried on with replacement, or, in the infinite case, without replacement.

21. Other Replacement Laws. The two cases just examined represent two extremes of orderly replacement laws. It has been shown in each case how the Carver functions can be used to express relations between the moments of the moments of the sample and the moments of the universe. It is possible to show how these functions are applicable to other replacement laws. We take, as an illustration, the case in which no replacements are made after each of the first $n - 1$ drawings, but just before the last drawing the $n - 1$ variates are replaced and mixed. I do not present here the detailed argument, but simply indicate that the appropriate value of $k_{p_1 \cdots p_s}$ is

$$\begin{aligned}k_{p_1 \cdots p_s} &= \frac{n^{(s)}}{N^{(s)}} + \frac{n-1}{N^{(s)}N} [(n-2)^{(s)} - n^{(s)} + (2^{p_1} + 2^{p_2} + \cdots + 2^{p_s})(n-2)^{(s-1)}] \quad \{28\}\end{aligned}$$

22. Different Frequency Laws. The distribution of variates may follow some known frequency law such as the normal, rectangular, binomial, Poisson, etc. In such a case, if the relations between the moments are known, it is possible to simplify the results.

Chapter IV. The Moments of the Mean

To illustrate the previous theory in a simple situation we consider the moments of the mean. Carver [25] has done this previously for the case of finite sampling without replacements, but he has taken the measures of the universe as deviations and has used the sample sum rather than the sample mean. O'Toole [26] has generalized Carver's work.

23. The Moments of the Mean. We have at once

$$\mu_1(lm_1) = \frac{1}{n} E(1) = \frac{1}{n} P_1(1) = \mu_1$$

$$\mu_2(lm_1) = \frac{1}{n^2} E(1)^2 = \frac{1}{n^2} [P_2(2) + P_{11}(1)(1)]$$

$$\mu_3(lm_1) = \frac{1}{n^3} E(1)^3 = \frac{1}{n^3} [P_3(3) + 3P_{21}(2)(1) + P_{111}(1)(1)(1)]$$

$$\begin{aligned} \mu_4(lm_1) = \frac{1}{n^4} E(1)^4 = \frac{1}{n^4} [P_4(4) + 4P_{31}(3)(1) + 3P_{22}(2)(2) + 6P_{211}(2)(1)(1) \\ + P_{1111}(1)^4] \end{aligned}$$

and

$$\mu_r(lm_1) = \frac{1}{n^r} \sum \binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}} P_{p_1^{\pi_1} \dots p_s^{\pi_s}} (P_1)^{\pi_1} \dots (P_s)^{\pi_s} \quad \{29\}$$

24. Moments About the Mean of the Sample Mean. Using $\{1\}$, we get

$$\bar{\mu}_2(lm_1) = \frac{1}{n^2} [P_2(2) + (P_{11} - P_1^2)(1)(1)]$$

$$\bar{\mu}_3(lm_1) = \frac{1}{n^3} [P_3(3) + 3(P_{21} - P_2 P_1)(2)(1) + (P_{111} - 3P_{11} P_1 + 2P_1^3)(1)^3]$$

$$\begin{aligned} \bar{\mu}_4(lm_1) = \frac{1}{n^4} [P_4(4) + 4(P_{31} - P_3 P_1)(3)(1) + 3P_{22}(2)(2) \\ + 3(P_{211} - 2P_{21} P_1 + P_2 P_1^2)(2)(1)(1) \\ + (P_{1111} - 4P_{111} P_1 + 6P_{11} P_1^2 - 3P_1^4)(1)^4] \quad \{30\} \end{aligned}$$

etc.

These formulas can be written in the notation of moments of the universe as

$$\bar{\mu}_2(^1m_1) = \frac{1}{n^2} [P_2 N \mu_2 + (P_{11} - P_1^2) N^2 \mu_1^2]$$

$$\bar{\mu}_3(^1m_1) = \frac{1}{n^3} [P_3 N \mu_3 + 3(P_{21} - P_2 P_1) N^2 \mu_2 \mu_1 + (P_{111} - 3P_{11} P_1 + 2P_1^3) N^3 \mu_1^3] \quad \{31\}$$

etc.

25. Moments of the Sample Mean When the Universe is Measured About its Mean. When $(1) = 0$, the formulas of section {23} become

$$\mu_1(m_1) = 0$$

$$\mu_2(m_1) = \frac{1}{n^2} P_1(\bar{2})$$

$$\mu_3(m_1) = \frac{1}{n^3} P_3(\bar{3})$$

$$\mu_4(m_1) = \frac{1}{n^4} [P_4(\bar{4}) + 3P_{22}(\bar{2})^2]$$

and

$$\mu_r(m_1) = \frac{1}{n^r} \sum \binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}} P_{p_1^{\pi_1} \dots p_s^{\pi_s}} (\bar{P}_1)^{\pi_1} \dots (\bar{P}_s)^{\pi_s} \quad \{32\}$$

where the $\sum$ holds for all partitions having no unit parts. In the language of moments {32} becomes

$$\mu_r(m_1) = \frac{1}{n^r} \sum \binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}} P_{p_1^{\pi_1} \dots p_s^{\pi_s}} N^{p_1^{\pi_1} \dots p_s^{\pi_s}} (\bar{\mu}_{p_1})^{\pi_1} \dots (\bar{\mu}_{p_s})^{\pi_s} \quad \{33\}$$

where the $\sum$ holds for all partitions of r having no unit parts.

26. Moments About the Mean of the Sample When the Universe is Measured From its Mean. Similarly, when $(1) = 0$, the results of section {24} become

$$\left. \begin{aligned} \bar{\mu}_2(m_1) &= \frac{1}{n^2} P_2(\bar{2}) = \frac{1}{n^2} P_2 N \bar{\mu}_2 \\ \bar{\mu}_3(m_1) &= \frac{1}{n^3} P_3(\bar{3}) = \frac{1}{n^3} P_3 N \bar{\mu}_3 \\ \bar{\mu}_4(m_1) &= \frac{1}{n^4} [P_4(\bar{4}) + 3P_{22}(\bar{2})^2] \\ &= \frac{1}{n^4} [P_4 N \bar{\mu}_4 + 3P_{22} N^2 \bar{\mu}_2^2] \\ \text{etc.} \end{aligned} \right\} \quad \{34\}$$

It is to be noticed that the values $\bar{\mu}_r(m_1)$ are equal to the values $\mu_r(m_1)$. This results from {4} and the fact that $\mu_1(m_1) = 0$. It should be noted also that $\bar{\mu}_r({}^1m_1) \approx \mu_r({}^1m_1)$ as $\mu_1({}^1m_1) \approx 0$.

27. Sampling Without Replacements. The formulas in sections 23-26 are general formulas which become more specific as given replacement laws are introduced. If the law is sampling without replacements, we recall that $P_1 = \rho_1$, $P_2 = \rho_1 - \rho_2$, $P_3 = \rho_1 - 3\rho_2 + 2\rho_3$, etc. when $\rho_s = \frac{n^{(s)}}{N^{(s)}}$. It is at once possible to write the appropriate formula. Thus

$$\begin{aligned}\bar{\mu}_3(m_1) &= \mu_3(m_1) = \frac{1}{n^3} P_{3, 3} \\ &= \frac{1}{n^3} [\rho_1 - 3\rho_2 + 2\rho_3] N \mu_3 = \frac{(N-n)(N-2n)}{n^2(N-1)(N-2)} \bar{\mu}_3. \quad \{35\}\end{aligned}$$

Now $\bar{\mu}_3 = 0$ in any symmetric universe, for example a normal or rectangular one, so $\bar{\mu}_3(m_1) = 0$.

28. Sampling With Replacements. In this case $P_{m_1 \dots m_r} = \frac{n^{(r)}}{N^r}$ and we have

$$\begin{aligned}\mu_1({}^1m_1) &= \mu_1 \\ \mu_2({}^1m_1) &= \frac{1}{n^2} [n\mu_3 + n(n-1)\mu_1^2] \\ \mu_3({}^1m_1) &= \frac{1}{n^3} [n\mu_3 + 3n(n-1)\mu_2\mu_1 + n(n-1)(n-2)\mu_1^3] \\ \mu_4({}^1m_1) &= \frac{1}{n^4} [n\mu_4 + 4n(n-1)\mu_3\mu_1 + 3n(n-1)\mu_2^2 \\ &\quad + 6n(n-1)(n-2)\mu_2\mu_1^2 + n^{(4)}\mu_1^4]\end{aligned}$$

and in general

$$\mu_r({}^1m_1) = \frac{1}{n^r} \sum \binom{1^r}{p_1^{\pi_1} \dots p_s^{\pi_s}} n^{(\rho)} (\mu_{p_1})^{\pi_1} \dots (\mu_{p_s})^{\pi_s} \quad \{36\}$$

and

$$\begin{aligned}\bar{\mu}_2({}^1m_1) &= \frac{1}{n^2} [n\mu_2 - n\mu_1^2] \\ \bar{\mu}_3({}^1m_1) &= \frac{1}{n^3} [n\mu_3 - 3n\mu_2\mu_1 + 2n\mu_1^3] \\ \bar{\mu}_4({}^1m_1) &= \frac{1}{n^4} [n\mu_4 - 4n\mu_3\mu_1 + 3n(n-1)\mu_2^2 - 6n(n-2)\mu_2\mu_1^2 + 3n(n-2)\mu_1^4]\end{aligned} \quad \{37\}$$

while

$$\begin{aligned}\bar{\mu}_2(m_1) &= \mu_2(m_1) = \frac{\bar{\mu}_2}{n} \\ \bar{\mu}_3(m_1) &= \mu_3(m_1) = \frac{\bar{\mu}_3}{n^2} \\ \bar{\mu}_4(m_1) &= \mu_4(m_1) = \frac{1}{n^3} [\bar{\mu}_4 + 3(n-1)\bar{\mu}_2^2] \\ &\text{etc.}\end{aligned}\quad \{38\}$$

29. Sampling With Replacements Before the Last Drawing Only. The values of $k_{p_1 \dots p_s}$ of section 21 determine the values of the P 's. Thus $P_2 = k_2 - k_{11} = \frac{n}{N} - \frac{n(n-1)}{N(N-1)} + \frac{2(n-1)}{N(N-1)}$ and $P_{11} = \frac{n(n-1)}{N(N-1)} - \frac{2(n-1)}{N^2(N-1)}$ so

$$\bar{\mu}_2(m_1) = \frac{1}{n^2} \left\{ \left[n - \frac{(n-1)(n-2)}{N-1} \right] \mu_2 + \frac{(n-1)(nN-2)}{N-1} \mu_1^2 \right\}. \quad \{39\}$$

$$\bar{\mu}_2(m_1) = \frac{1}{n^2} \left\{ \left[n - \frac{(n-1)(n-2)}{N-1} \right] \bar{\mu}_2 \right\}. \quad \{40\}$$

30. Different Frequency Laws. As indicated in section 22, the frequency distributions of the parent may be characterized by some moment relationship. This relationship can be inserted and the resulting formula simplified. For example, if the law of the formation of the universe is that of the hypergeometric series [25; 113]

$$\bar{\mu}_n = pq[q^{n-1} + (-1)^n p^{n-1}], \quad \{41\}$$

we have

$$\left. \begin{aligned}\bar{\mu}_2(m_1) &= \frac{P_2}{n^2} Npq \\ \bar{\mu}_3(m_1) &= \frac{P_3}{n^3} Npq(q^2 - p^2) \\ \bar{\mu}_4(m_1) &= \frac{1}{n^4} [P_4 Npq(q^3 + p^3) + 3P_{22} N^2 p^2 q^2] \\ &\text{etc.}\end{aligned} \right\} \quad \{42\}$$

Where the values of P_2, P_3, P_4 are to be inserted according to the replacement law which is used in forming the samples. The results for sampling without replacement agree with those given by Pearson [1].

31. Moments of the Sample Sum. We might use the sum of the items in the sample instead of the sample mean. For example

$$\mu_2(1) = E(1)^2 = n^2 E(m_1)^2 = n^2 \mu_2(m_1).$$

The results would parallel the results above except that n^r in the denominator would be eliminated. It is the sample sum which is used in Carver's article [25] and this should be noted in comparing results.

Chapter V. The Mean and Variance of the Variance

As a further illustration of the use of the Carver functions there are presented in this chapter formulas for the mean of the variance and the variance of the variance.

32. The Mean of the Variance.

$$\begin{aligned}\mu_1({}^1\bar{m}_2) &= E \left[\frac{(2)}{n} - \frac{(1)(1)}{n^2} \right] \\ &= \frac{P_1}{n} (2) - \frac{P_2(2) + P_{11}(1)^2}{n^2} \\ &= \frac{1}{n^2} [(nP_1 - P_2)(2) - P_{11}(1)^2]\end{aligned}\quad \{43\}$$

and

$$\mu_1(\bar{m}_2) = \frac{1}{n^2} (nP_1 - P_2)N\bar{\mu}_2. \quad \{44\}$$

When sampling is with replacements $P_1 = P_2 = \frac{n}{N}$ and we get the well known

$$\mu_1(\bar{m}_2) = \frac{(n-1)}{n} \bar{\mu}_2 \quad \{45\}$$

while when sampling is without replacements, we have the well known

$$\mu_1(\bar{m}_2) = \frac{\left(1 - \frac{1}{n}\right)\bar{\mu}_2}{1 - \frac{1}{N}}. \quad \{46\}$$

33. The Second Moment of the Variance.

$$\mu_2({}^1\bar{m}_2) = E \left[\frac{(2)^2}{n^2} - \frac{2(2)(1)(1)}{n^3} + \frac{(1)^4}{n^4} \right]$$

becomes

$$\begin{aligned}\mu_2({}^1\bar{m}_2) &= \left(\frac{P_2}{n^2} - \frac{2P_3}{n^3} + \frac{P_4}{n^4} \right) (4) - 4 \left(\frac{P_{21}}{n^3} - \frac{P_{31}}{n^4} \right) (3)(1) \\ &+ \left(\frac{P_{11}}{n^2} - \frac{2P_{21}}{n^3} + \frac{3P_{22}}{n^4} \right) (2)(2) - 2 \left(\frac{P_{111}}{n^3} - \frac{3P_{211}}{n^4} \right) (2)(1)(1) + P_{1111}(1)^4\end{aligned}\quad \{47\}$$

and

$$\mu_2(\bar{m}_2) = \left(\frac{P_2}{n^2} - \frac{2P_3}{n^3} + \frac{P_4}{n^4} \right) (\bar{4}) + \left(\frac{P_{11}}{n^2} - \frac{2P_{21}}{n^3} + \frac{3P_{22}}{n^4} \right) (\bar{2})^2. \quad \{48\}$$

These of course can be written in terms of moments of the variance.

34. The Variance of the Variance. Since $\bar{\mu}_2({}^1\bar{m}_2) = \mu_2({}^1\bar{m}_2) - \mu_1^2({}^1\bar{m}_2)$, we have

$$\begin{aligned} \bar{\mu}_2({}^1\bar{m}_2) &= \left(\frac{P_2}{n^2} - \frac{2P_3}{n^3} + \frac{P_4}{n^4} \right) (4) - 4 \left(\frac{P_{21}}{n^3} - \frac{P_{31}}{n^4} \right) (3)(1) \\ &+ \left[\frac{P_{11}}{n^2} - \frac{2P_{21}}{n^3} + \frac{3P_{22}}{n^4} - \left(\frac{P_1}{n} - \frac{P_2}{n^2} \right)^2 \right] (2)(2) \\ &- 2 \left[\frac{P_{111}}{n^3} - \frac{3P_{211}}{n^4} - \frac{P_1 P_{21}}{n^3} + \frac{P_2 P_{11}}{n^4} \right] (2)(1)(1) + \frac{P_{1111} - P_{11}^2}{n^4} (1)^4. \quad \{49\} \end{aligned}$$

Formula {49} may also be written as

$$\begin{aligned} \bar{\mu}_2({}^1\bar{m}_2) &= \frac{1}{n^4} \{ (n^2 P_2 - 2n P_3 + P_4) N \mu_4 - 4(n P_{21} - P_{31}) N^2 \mu_3 \mu_1 \\ &+ (n^2 P_{11} - 2n P_{21} + 3P_{22} - n^2 P_1^2 + 2n P_1 P_2 - P_2^2) N^2 \mu_2^2 \\ &- 2(n P_{111} - 3P_{211} - n P_1 P_{11} + P_2 P_{11}) N^3 \mu_2 \mu_1^2 + (P_{1111} - P_{11}^2) N^4 \mu_1^4 \}. \quad \{50\} \end{aligned}$$

Formulas {49} and {50} are not expressed in terms of deviations of the variates. Neither do they assume any particular replacement law nor any particular type of universe.

In case the universe is measured about its mean we can write at once, by placing $(1) = 0$ in {49}

$$\begin{aligned} \bar{\mu}_2(\bar{m}_2) &= \left(\frac{P_2}{n^2} - \frac{2P_3}{n^3} + \frac{P_4}{n^4} \right) (\bar{4}) \\ &+ \left[\left(\frac{P_{11}}{n^2} - \frac{2P_{21}}{n^3} + \frac{3P_{22}}{n^4} \right) - \left(\frac{P_1}{n} - \frac{P_2}{n^2} \right)^2 \right] (\bar{2})(\bar{2}) \quad \{51\} \end{aligned}$$

and

$$\begin{aligned} \bar{\mu}_2(\bar{m}_2) &= \frac{1}{n^4} \{ (n^2 P_2 - 2n P_3 + P_4) N \bar{\mu}_4 + (n^2 P_{11} - 2n P_{21} + 3P_{22} - n^2 P_1^2 \\ &+ 2n P_1 P_2 - P_2^2) N \bar{\mu}_2^2 \}. \quad \{52\} \end{aligned}$$

35. Sampling Without Replacements. Using the P 's as defined by sampling without replacements, it appears that the coefficient of the μ_4 term

$$\left(\frac{P_2}{n^2} - \frac{2P_3}{n^3} + \frac{P_4}{n^4} \right) N = \frac{N}{n^3} \frac{(N-n)(n-1)(Nn-N-n-1)}{(N-1)(N-2)(N-3)} \quad \{53\}$$

agrees with that given by Neyman [18; 477], Tchouproff [15; 660], Pepper [24; 234], Carver [25; 270]. Also the coefficient of the μ_2^2 term

$$\begin{aligned} \frac{P_{11}}{n^2} - \frac{2P_{21}}{n^3} + \frac{3P_{22}}{n^4} - \left(\frac{P_1}{n} - \frac{P_2}{n^2} \right)^2 N^2 \\ = - \frac{N(N-n)(n-1)(N^2n - 3N^2 + 6N - 3n - 3)}{n^3(N-1)^2(N-2)(N-3)} \end{aligned} \quad \{54\}$$

agrees with that of the above authors.

As far as the author is aware, no one has written the coefficients of $\mu_3\mu_1$, $\mu_2\mu_1^2$, and μ_1^4 in the formula for $\bar{\mu}_2(^1\bar{m}_2)$.

The coefficient of $\mu_3\mu_1$ is

$$-4 \left(\frac{P_{21}}{n^3} - \frac{P_{31}}{n^4} \right) N^2 = \frac{-4N(n-1)(N-n)(Nn - N - n - 1)}{n^3(N-1)(N-2)(N-3)}. \quad \{55\}$$

The coefficient of $\mu_2\mu_1^2$ is

$$\begin{aligned} -2 \left(\frac{P_{111}}{n^3} - \frac{3P_{211}}{n^4} - \frac{P_1P_{11}}{n^3} + \frac{P_2P_{11}}{n^4} \right) N^3 \\ = \frac{4N^2(n-1)(N-n)[(2n-3)N - 3(n-1)]}{n^3(N-1)^2(N-2)(N-3)} \end{aligned} \quad \{56\}$$

while the coefficient of μ_1^4 is

$$\left(\frac{P_{1111}}{n^4} - \frac{P_{11}P_{11}}{n^4} \right) N^4 = - \frac{2N^2(n-1)(N-n)[(2n-3)N - 3(n-1)]}{n^3(N-1)^2(N-2)(N-3)}. \quad \{57\}$$

It is possible with some algebraic manipulation to use the P functions to express the coefficients of the moments as functions of N and n . The suggestion here is that such algebraic work is unnecessary since the left members of {53} . . . {57} are as easily handled in an actual problem as the right hand members. It is possible to compute the coefficients from the ρ 's and the P 's without writing explicit expansions in terms of N and n . Besides the formulas involving N and n are so lengthy that algebraic errors are apt to occur. The use of Carver functions is further advocated because the same basic formulas are applicable to all types of sampling and because the tables of Chapter I of Part I are directly applicable.

36. Sampling With Replacements. If $P_{m_1 \dots m_r} = \frac{n^{(r)}}{N^{(r)}}$, the coefficient of μ_4 is $\frac{1}{n^4} [(n(n-1)^2)] = \frac{(n-1)^2}{n^3}$ while the coefficient of μ_2^2 is

$$\frac{1}{n^4} [(n^2 - 2n + 3)n(n-1) - (n^2 - 2n + 1)n^2] = \frac{(n-1)(3-n)}{n^3}.$$

Then {52} becomes

$$\bar{\mu}_2(\bar{m}_2) = \frac{1}{n^3} [(n-1)^2 \bar{\mu}_4 - (n-1)(n-3) \bar{\mu}_2^2]. \quad \{58\}$$

The formula for $\bar{\mu}_2(^1\bar{m}_2)$ becomes

$$\begin{aligned} \bar{\mu}_2(^1\bar{m}_2) = \frac{1}{n^3} [(n-1)^2 \mu_4 - 4(n-1)^2 \mu_3 \mu_1 - (n-1)(n-3) \mu_2^2 \\ + 4(2n-3)(n-1) \mu_2 \mu_1^2 - 2(2n-3)(n-1) \mu_1^4]. \quad \{59\} \end{aligned}$$

Now {58} can be written in terms of semi-invariants by the use of $\bar{\mu}_4 = \lambda_4 + 3\lambda_2^2$ and $\bar{\mu}_2 = \lambda_2$ so

$$\bar{\mu}_2(\bar{m}_2) = \frac{1}{n^3} [(n-1)^2 \lambda_4 + 2n(n-1) \lambda_2^2].$$

See [B'; 209], [22; 57].

37. Different Distribution Laws. Given frequency laws can be inserted. Thus {44} becomes

$$\mu_1(\bar{m}_2) = \frac{1}{n^2} (nP_1 - P_2)pq \quad \text{if the} \quad \bar{\mu}_2 = pq$$

while {52} becomes, if $\bar{\mu}_2 = pq$ and $\bar{\mu}_4 = pq(q^3 + p^3)$

$$\begin{aligned} \bar{\mu}_2(\bar{m}_2) = \frac{N}{n^4} (n^2 P_2 - 2nP_3 + P_4)pq(q^3 + p^3) \\ + \frac{N^2}{n^4} (n^2 P_{11} - 2nP_{21} + 3P_{22} - n^2 P_1^2 + 2nP_2 P_1 - P_2^2)p^2 q^2. \quad \{60\} \end{aligned}$$

Other frequency laws can be inserted similarly.

Chapter VI. Tabular Presentation of Formulas.

It is the purpose of this dissertation to show how the P functions can be used in finite sampling rather than to present an exhaustive list of formulas. The specific formulas of the two previous chapters are derived, primarily, for illustrative purposes. The implication is that other formulas may be derived similarly.

However, it is possible to present, implicitly in tabular form, a number of formulas. In this chapter there are presented formulas involving moments of weight equal to or less than 6.

38. The formulas of weight 2.

$$\left. \begin{aligned} \mu_1(^1\bar{m}_2) &= \left(\frac{P_1}{n} - \frac{P_2}{n^2} \right) (2) - \frac{P_{11}}{n^2} (1)(1) \\ \mu_2(^1m_1) &= \left[\frac{P_2}{n^2} (2) + \frac{P_{11}}{n^2} (1)^2 \right] \end{aligned} \right\} \quad \{61\}$$

can be written in tabular form as

	2	11		2	11
2	P_1			$\frac{1}{n}$	
11	P_2	P_{11}		$-\frac{1}{n^2}$	$\frac{1}{n^2}$

with little effort. The first entries in the top row indicate the power sums of the universe, while the columnar entries indicate the moments of the sample. Now

$$\mu_1(\bar{m}_2) = E \left[\frac{(2)}{n} - \frac{(1)(1)}{n^2} \right]$$

and

$$\mu_2(m_1) = E \left[\frac{(1)(1)}{n^2} \right].$$

The coefficients of the power sums in the expansion of $\bar{m}$ are entered in the right hand part of the table. Thus, under 2, there appear the entries $\frac{1}{n}$ and $-\frac{1}{n^2}$. These when multiplied by the power sums as indicated on the left, give $\bar{m}_2 = \frac{(2)}{n} - \frac{(1)(1)}{n^2}$. Similarly $m_1^2 = \frac{(1)(1)}{n^2}$.

Now the expected value is given by the proper P function expansion. The left hand portion of the table, which is the same as the P function table of Chapter I of Part I, gives such expansions. Thus the coefficient of (2) in $E(m_2)$ is $\frac{P_1}{n} - \frac{P_2}{n^2}$, while the coefficient of (1)(1) is $-\frac{P_{11}}{n^2}$. Hence the complete formula is

$$\mu_1(\bar{m}_2) = \left(\frac{P_1}{n} - \frac{P_2}{n^2} \right) (2) - \frac{P_{11}}{n^2} (1)(1)$$

as indicated above.

39. The Formulas of Weight 3. Similarly the table

	3	21	111		3	21	111
3	P_1				$\frac{1}{n}$		
21	P_2	P_{11}			$-\frac{3}{n^2}$	$\frac{1}{n^2}$	
111	P_3	$3P_{21}$	P_{111}		$\frac{2}{n^3}$	$-\frac{1}{n^3}$	$\frac{1}{n^3}$

can be used to give the formulas

$$\mu_1(\bar{m}_3) = \left(\frac{P_1}{n} - \frac{3P_2}{n^2} + \frac{2P_3}{n^3} \right) (\mathbf{3}) - 3 \left(\frac{P_{11}}{n^2} - \frac{2P_{21}}{n^3} \right) (\mathbf{2})(\mathbf{1}) + 2 \frac{P_{111}}{n^3} (\mathbf{1})^3 \quad \{62\}$$

$$\mu_{11}(\bar{m}_2, m_1) = \left(\frac{P_2}{n^2} - \frac{P_3}{n^3} \right) (\mathbf{3}) + \left(\frac{P_{11}}{n^2} - \frac{3P_{21}}{n^3} \right) (\mathbf{2})(\mathbf{1}) - \frac{2P_{111}}{n^3} (\mathbf{1})^3 \quad \{63\}$$

$$\mu_3(m_1) = \frac{P_3}{n^3} (\mathbf{3}) + \frac{3P_{21}}{n^3} (\mathbf{2})(\mathbf{1}) + \frac{P_{111}}{n^3} (\mathbf{1})^3. \quad \{64\}$$

In case we wish to express the results in terms of moments about the mean, $(\mathbf{1}) = 0$, and we have

$$\mu_1(\bar{m}_3) = \left(\frac{P_1}{n} - \frac{3P_2}{n^2} + \frac{2P_3}{n^3} \right) (\bar{\mathbf{3}}) \quad \{65\}$$

$$\mu_{11}(\bar{m}_2, m_1) = \left(\frac{P_2}{n^2} - \frac{P_3}{n^3} \right) (\bar{\mathbf{3}}) \quad \{66\}$$

$$\mu_3(m_1) = \frac{P_3}{n^3} (\bar{\mathbf{3}}) \quad \{67\}$$

so that

$$\mu_1(\bar{m}_3) = \left(\frac{P_1}{n} - \frac{3P_2}{n^2} + \frac{2P_3}{n^3} \right) N\bar{\mu}_3 \quad \{68\}$$

$$\mu_{11}(\bar{m}_2, m_1) = \left(\frac{P_2}{n^2} - \frac{P_3}{n^3} \right) N\bar{\mu}_3 \quad \{69\}$$

$$\mu_3(m_1) = \frac{P_3}{n^3} N\bar{\mu}_3. \quad \{70\}$$

The insertion of specific sampling laws gives the specific results of earlier authors.

40. The Tabular Forms. It is further evident that the power of n in the denominator is equal to the sum of the subscripts of the Carver function above it. We might utilize this knowledge and write in the right hand part of the table the numerators of the entries in the tables above. The table of weight of 3 would then appear as

	3	21	111		3	21	111
3	P_1				1		
21	P_2	P_{11}			-3	1	
111	P_3	$3P_{21}$	P_{111}		2	-1	1

and it is possible to read {62}, {63}, {64}, {65}, {66}, and {67} directly from it.

TABLE I

$w = 2$

	$\varnothing$	11		2	11
2	P_1			1	
11	P_2	P_{11}		-1	1

$w = 3$

	$\varnothing$	21	1		3	21	1
3	P_1				1		
21	P_2	P_{11}			-3	1	
1	P_3	$3P_{21}$	P_{111}		2	-1	1

$w = 4$

	$\varnothing$	4	31	$\varnothing\varnothing$	211	1111		4	31	22	211	1111
4	P_1							1				
31	P_2	P_{11}						-4	1			
22	P_3		P_{11}							1		
211	P_4	$2P_{21}$	P_{21}	P_{111}				6	-3	-2	1	
1111		$4P_{31}$	$3P_{32}$	$6P_{211}$	P_{1111}			-3	2	1	-1	1

$w = 5$

	$\varnothing$	41	$\varnothing\varnothing$	311	221	2111	1^5	5	41	32	311	221	2111	1^5
5	P_1							1						
41	P_2	P_{11}						-5	1					
32	P_3		P_{11}							1				
311	P_4	$2P_{21}$	P_{21}	P_{111}				10	-4	-1	1			
221	P_5	P_{21}	$2P_{21}$		P_{111}					-3		1		
2111		$3P_{31}$	$P_{31} + 3P_{32}$	$3P_{211}$	$3P_{211}$	P_{1111}		-10	6	5	-3	-2	1	
11111		$5P_{41}$	$10P_{32}$	$10P_{311}$	$15P_{221}$	$10P_{2111}$		4	-3	-2	2	1	1	1

$$w = 6$$

	6	51	42	33	411	321	222	3111	2 ² 1 ²	2 1 ⁴	1 ⁶	6	51	42	33	411	321	222	3111	2 ² 1 ²	2 1 ⁴	1 ⁶
6	P_1											1										
51	P_2	P_{11}										-6	1									
42	P_2		P_{11}											1								
33	P_2			P_{11}											1							
411	P_3	$2P_{21}$	P_{31}		P_{111}							15	-5	-1		1						
321	P_3	P_{21}	P_{31}	P_{21}		P_{111}								-4	-6		1					
222	P_3		$3P_{21}$				P_{111}											1				
3111	P_4	$3P_{31}$	$3P_{22}$	P_{31}	$3P_{211}$			P_{1111}				20	10	4	4	-4	-1		1			
2211	P_4	$2P_{31}$	$P_{22} + 2P_{31}$	$2P_{22}$	P_{211}	$4P_{211}$	P_{211}		P_{1111}					6	9		-3	-3		1		
21 ⁴	P_5	$4P_{41}$	$P_{41} + 6P_{32}$	$4P_{32}$	$6P_{311}$	$4P_{311} + 12P_{221}$	$3P_{221}$	$4P_{2111}$	$6P_{2111}$	P_{11111}		-15	-10	-9	-12	6	5	3	-3	-2	1	
1 ⁶	P_6	$6P_{51}$	$15P_{42}$	$10P_{33}$	$15P_{411}$	$60P_{321}$	$15P_{222}$	$20P_{3111}$	$45P_{2211}$	$15P_{2111}$	P_{111111}	5	4	3	4	-3	-2	-1	2	1	-1	1

The tables of weight $W = 2, 3, 4, 5, 6$ are given in Table I. The right hand partitions not involving unit parts are underscored as these indicate the columns which should be used if the universe is measured about its mean. As an illustration we write from Table I the value of $\mu_2(\bar{m}_2)$. We get

$$\mu_2(\bar{m}_2) = \left(\frac{P_2}{n^2} - \frac{2P_3}{n^3} + \frac{P_4}{n^4} \right) N\bar{\mu}_4 + \left(\frac{P_{11}}{n^2} - \frac{2P_{21}}{n^3} + \frac{3P_{22}}{n^4} \right) N\bar{\mu}_2^2$$

as previously indicated.

The same tabular scheme can be used to write formulas of weight greater than 6.

41. Moments of Other Sample Moment Functions. It is possible to use a similar tabular scheme when we wish to find the moments of other sample moment functions. We define

$$l_2 = \frac{(2)}{n} - \frac{(1)(1)}{n^2}$$

$$l_3 = \frac{(3)}{n} - \frac{3(2)(1)}{n^2} + \frac{2(1)^3}{n^3}$$

$$l_4 = \frac{(4)}{n} - \frac{4(3)(1)}{n^2} - \frac{3(2)(2)}{n^2} + \frac{12(2)(1)(1)}{n^3} - \frac{6(1)^4}{n^4}$$

and, in general,

$$l_n = \sum (-1)^{\rho-1} (\rho-1)! \left(\begin{matrix} 1^r \\ p_1^{\pi_1} \dots p_s^{\pi_s} \end{matrix} \right) \frac{(p_1)^{\pi_1} \dots (p_s)^{\pi_s}}{n^\rho}. \quad \{71\}$$

The formulas of weight 5 are given by Table II.

TABLE II

	δ	41	$\underline{32}$	311	221	2111	1^5		-5	41	32	311	221	21^3	1^6
5	P_1								1						
41	P_2	P_{11}							-5	1					
32	P_2		P_{11}						-10		1				
311	P_3	$2P_{21}$	P_{21}	P_{111}					20	-4	-1	1			
221	P_3	P_{21}	$2P_{21}$		P_{111}				30	-3	-3		1		
2111	P_4	$3P_{31}$	$P_{31} + 3P_{22}$	$3P_{211}$	$3P_{211}$	P_{1111}			-60	12	5	-3	-2	1	
11111	P_5	$5P_{41}$	$10P_{32}$	$10P_{311}$	$15P_{221}$	$10P_{2111}$	P_{11111}		24	-6	-2	2	1	-1	1

Thus for example

$$\begin{aligned} {}^1\mu_{11}(l_4, l_1) = & \left(\frac{P_2}{n^2} - \frac{7P_3}{n^3} + \frac{12P_4}{n^4} - \frac{6P_5}{n^5} \right) N\bar{\mu}_5 \\ & + \left(-\frac{10P_{21}}{n^3} + \frac{12P_{31}}{n^4} + \frac{36P_{22}}{n^4} - \frac{60P_{32}}{n^5} \right) N^2\bar{\mu}_3\bar{\mu}_2. \quad \{72\} \end{aligned}$$

If all the entries in the right hand part of Table I, except the unit terms in the main diagonal, are placed equal to 0, the tables can be used to give the moment function of the m_a . Thus, when $w = 3$,

$$\mu_1({}^1m_3) = \frac{P_1}{n} (3) \quad \{73\}$$

$${}^1\mu_{11}(m_2, m_1) = \frac{P_2}{n^2} (3) + \frac{P_{11}}{n^2} (2)(1) \quad \{74\}$$

$$\mu_3({}^1m_1) = \frac{P_3}{n^3} (3) + \frac{3P_{21}}{n^3} (2)(1) + \frac{P_{111}}{n^3} (1)^3 \quad \{75\}$$

and

$$\mu_1(m_3) = \frac{P_1}{n} N\bar{\mu}_3 \quad \{76\}$$

$$\mu_{11}(m_2, m_1) = \frac{P_2}{n^2} N\bar{\mu}_3 \quad \{77\}$$

$$\mu_3(m_1) = \frac{P_3}{n^3} N\bar{\mu}_3. \quad \{78\}$$

42. Other Moment Functions. The tables give such formulas as $\mu_r(\bar{m}_a)$, $\mu_{r_1r_2}(\bar{m}_a, \bar{m}_b)$, etc. If formulas for $\bar{\mu}_r(\bar{m}_a)$, $\bar{\mu}_{r_1r_2}(\bar{m}_a, \bar{m}_b)$ etc., are needed, it is necessary to go through the usual work of changing from moments to moments about the mean.

Let us derive a general formula for the correlation of the mean and the variance as an illustration of the use of the tabular formulas. By definition

$$r_{11}(\bar{m}_2, m_1) = \frac{\bar{\mu}_{11}(\bar{m}_2, m_1)}{[\bar{\mu}_{20}(\bar{m}_2, m_1)\bar{\mu}_{02}(\bar{m}_2, m_1)]^{\frac{1}{2}}}. \quad \{79\}$$

Now

$$\bar{\mu}_{11}(\bar{m}_2, m_1) = \mu_{11}(\bar{m}_2, m_1)$$

$$\bar{\mu}_{20}(\bar{m}_2, m_1) = \mu_2(\bar{m}_2) - \mu_1^2(\bar{m}_2)$$

$$\bar{\mu}_{02}(\bar{m}_2, m_1) = \mu_2(m_1) - \mu_1^2(m_1) = \mu_2(m_1).$$

Some of these values have appeared earlier in this paper. Without using the earlier results, we find from Table I

$$\begin{aligned}\mu_{11}(\bar{m}_2, m_1) &= \left(\frac{P_2}{n^2} - \frac{P_3}{n^3}\right) N \bar{\mu}_3 \\ \mu_2(\bar{m}_2) &= \left(\frac{P_2}{n^2} - \frac{2P_3}{n^3} + \frac{P_4}{n^4}\right) N \bar{\mu}_4 + \left(\frac{P_{11}}{n^2} - \frac{2P_{21}}{n^3} + \frac{3P_{22}}{n^4}\right) N^2 \bar{\mu}_2^2 \\ \mu_1(\bar{m}_2) &= \left(\frac{P_1}{n} - \frac{P_2}{n^2}\right) N \bar{\mu}_2 \\ \mu_2(m_1) &= \frac{P_2}{n^2} N \bar{\mu}_2.\end{aligned}$$

Hence {79} becomes

$$r_{11}(\bar{m}_2, m_1) = \frac{(nP_2 - P_3)\bar{\mu}_3}{[(n^2P_2^2 - 2nP_2P_3 + P_2P_4)\bar{\mu}_4\bar{\mu}_2 - (n^2P_2P_{11} - 2nP_2P_{21} + 3P_2P_{22} - n^2P_2P_1^2 + 2nP_2^2P_1 - P_2^3)N\bar{\mu}_2^3]} \quad \{80\}$$

Formula {80} gives the correlation between the variance and the mean no matter what the law of replacement. If the universe is symmetric, $\bar{\mu}_3 = 0$ and $r_{11}(\bar{m}_2, m_1) = 0$.

The usual special cases may be obtained. When replacements are made, {80} becomes at once

$$r_{11}(\bar{m}_2, m_1) = \frac{(n-1)\bar{\mu}_3}{[(n-1)\bar{\mu}_4\bar{\mu}_2 - (3-n)\bar{\mu}_2^3]} \quad \{81\}$$

as indicated by Pepper [24; 246].

When no replacements are made {80} reduces to results previously given by Neyman [18; 489] and Pepper [24; 245].

43. Conclusion. The theory presented here is capable of generalization in many ways. For example, application to multivariate distributions readily follows. However an attempt has been made in this dissertation to emphasize the essence of the method. Illustrations have been chosen to indicate its inherent generality.

It should be stated, finally, that the aim of this dissertation is not primarily to provide a list of sampling formulas, but rather to provide a method by which the desired sampling formula may be derived without too much algebraic work.

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